1. MATH547 HOMEWORK 1

Topic:	Math@UNC environment
Post date:	May 14, 2020
Due date:	May 15, 2020

1.1. Background

This homework is meant as a tutorial on matrix computations, in particular:

- vector addition and scaling;
- vector norms;
- inner products;
- matrix-vector products;
- matrix-matrix products.

1.2. Theoretical questions

1.2.1. Vector addition and scaling

Problem. Consider n random walks on the real line with m unit steps starting from the origin. What is the mean distance and mean squared distance from the origin?

Answer. Denote distance by d, and the random walk steps as $u_1, u_2, \ldots$

$$d = u_1 + u_2 + \cdots + u_m = \sum_{i=1}^{m} u_i, u_i \in \{-1, 1\}.$$

Denote by $\mu(d)$ the average distance over n walks which is the sum of the average displacement in each step

$$\mu(d) = \sum_{i=1}^{m} \mu(u_i)$$

Assuming that both left and right steps are equiprobable, $\mu(u_i) = 0$ and $\mu(d) = 0$.
The squared distance is

$$d^2 = (u_1 + u_2 + \cdots + u_m)^2 = \sum_{i=1}^{m} u_i^2 + 2 \sum_{i=1}^{m} \sum_{j=i+1}^{m} u_i u_j$$

The average is

$$\mu(d^2) = \sum_{i=1}^{m} \mu(u_i^2) + 2 \sum_{i=1}^{m} \sum_{j=i+1}^{m} \mu(u_i u_j).$$

Since $u_i^2 = 1$ the average is $\mu(u_i^2) = 1$. The possible values of $u_i u_j = \pm 1$ with equal probability $\mu(u_i u_j) = 0$. The average squared distance is

$$\mu(d^2) = m.$$

$$\boldsymbol{A} \in \mathbb{R}^{m \times n}, m \gg 1, n$$

```
octave] m=100; n=500; U=2*(rand(m,n)<0.5)-1; mean(mean(U))
```

```
ans = -0.0063600
```

```
octave] function md=meand(m,n)
          U=2*(rand(m,n)<0.5)-1;
          md=mean(mean(U));
        end
```

```
octave] Nmax=1000; md=zeros(50,1);
        for k=1:50
          md(k) = meand(m,100*k);
        end;
```

```
octave] plot(100:100:5000,abs(md),'o')
```

```
octave] pwd
```

```
ans = /home/student/courses/MATH547ML
```

```
octave] mkdir homework
```

```
ans = 1
```

```
octave] cd homework
```

```
octave] pwd
```

```
ans = /home/student/courses/MATH547ML/homework
```

```
octave] mkdir hw01;
```

```
octave] cd hw01
```

```
octave] print -deps hw01fig01.eps
```

```
octave] pwd
```

```
ans = /home/student/courses/MATH547ML/homework/hw01
```

```
octave] ls
```

```
hw01fig01.eps  untitled.ofig
```

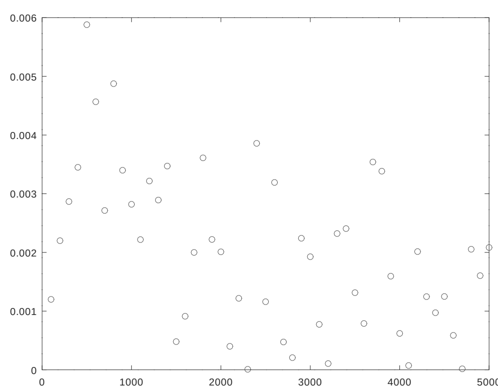

Figure 1. Experiment on statsitc to evaluate mean of random walk distance $\mu(d)$

1.2.2. Norms in E_m

Problem. Show that for $a_1, a_2 > 0$, $f: \mathbb{R}^2 \to \mathbb{R}_+$ defined by

$$f(\boldsymbol{x}) = a_1 x_1^2 + a_2 x_2^2,$$

is a norm in E_2.

Answer. Let $y_1 = \sqrt{a_1}\, x_1$, $y_2 = \sqrt{a_2}\, x_2$, or in vector form $f(\boldsymbol{x}) = g(\boldsymbol{y}) = y_1^2 + y_2^2$.

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}, \boldsymbol{A} = \begin{bmatrix} \sqrt{a_1} & 0 \\ 0 & \sqrt{a_2} \end{bmatrix},$$

Since $g(\boldsymbol{y})$ does not satisfy scaling property $\|a\boldsymbol{u}\| = |a|\,\|\boldsymbol{u}\|$, neither if f a norm. However $f = \sqrt{g}$ would be.

1.2.3. Norms in $\boldsymbol{E_m}$

Problem. Show that for $a_1, a_2 > 0$, $f: \mathbb{R}^2 \to \mathbb{R}_+$ defined by

$$f(\boldsymbol{x}) = a_1 x_1^2 - a_2 x_2^2$$

is not a norm in E_2.

Answer. No $f(\boldsymbol{x}) = 0$ for $\boldsymbol{x} = (1, \sqrt{a_1/a_2}) \neq \boldsymbol{0}$

1.2.4. Inner product in $\boldsymbol{E_m}$

Problem. Show that for $a_1, a_2 > 0$, $s: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$s(\boldsymbol{x}, \boldsymbol{y}) = a_1 x_1 y_1 + a_2 x_2 y_2,$$

is a scalar product in E_2.

Answer. Recall that

$$t(\boldsymbol{x}, \boldsymbol{y}) = [\ x_1 \ \ x_2\ ][\ \begin{matrix} y_1 \\ y_2 \end{matrix}\ ] = x_1 y_1 + x_2 y_2$$

is a scalar product. From

$$s(\boldsymbol{x}, \boldsymbol{y}) = t(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{y}) = t(\boldsymbol{x}, \boldsymbol{A}\boldsymbol{y}), \text{with} \boldsymbol{A} = \left[\begin{array}{cc} a_1 & 0 \\ 0 & a_2 \end{array}\right],$$

and fact that t is a scalar product, deduce s is also a scalar product.

1.2.5. Inner product in $\boldsymbol{E_m}$

Problem. Show that for $a_1, a_2 > 0$, $s: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$s(\boldsymbol{x}, \boldsymbol{y}) = a_1 x_1 y_1 - a_2 x_2 y_2,$$

is not a scalar product in E_2.

Answer. From $s(\boldsymbol{x}, \boldsymbol{x}) = a_1 x_1^2 - a_2 x_2^2$ notice that $s(\boldsymbol{x}, \boldsymbol{x})$ can be negative, hence s is not a scalar product.

1.2.6. Matrix-matrix products

Problem. Find examples of matrices for which $\boldsymbol{AB} = \boldsymbol{BA}$, and $\boldsymbol{CD} \neq \boldsymbol{DC}$.

Answer.

```
octave] A=[2 1; 1 2]; B=[2 0; 0 2]; A*B-B*A
```

```
ans =

   0   0
   0   0
```

```
octave] C=[2 1; -1 1]; D=[0 -2; 1 -1]; C*D-D*C
```

```
ans =

  -1  -3
  -2   1
```

```
octave]
```

```
Octave]
```

1.3. Linear functionals and mappings in analysis of EEG data

Electroencephalograms (EEGs) are recordings of the electric potential on cranium skin. Research on brain activity uses EEGs to determine specific activity patterns in the brain. For example, epileptic seizures have a distinctive EEG signature. EEG data can be loaded from the course data directory.

```
octave] load /home/student/courses/MATH547ML/data/eeg/eeg;
```

```
octave] data=EEG.data'; [m n]=size(data); disp([m n]);
```

```
   30504     32
```

```
octave]
```

There are $n = 32$ sensor recordings with $m = 30504$ components each. A first common operation is scaling of the data.

```
octave] pdata=data./max(data)+meshgrid(0:n-1,0:m-1);
```

```
octave] hold on;
        for j=1:n
          plot(pdata(:,j));
        end;
        hold off;
```

```
octave] cd /home/student/courses/MATH547ML/homework/hw01
```

```
octave] print -deps eeg.eps;
```

```
GL2PS info: OpenGL feedback buffer overflow
warning: gl2ps_renderer::draw: retrying with buffer size: 8.4E+06 B
GL2PS info: OpenGL feedback buffer overflow
warning: gl2ps_renderer::draw: retrying with buffer size: 1.7E+07 B
```

```
octave]
```

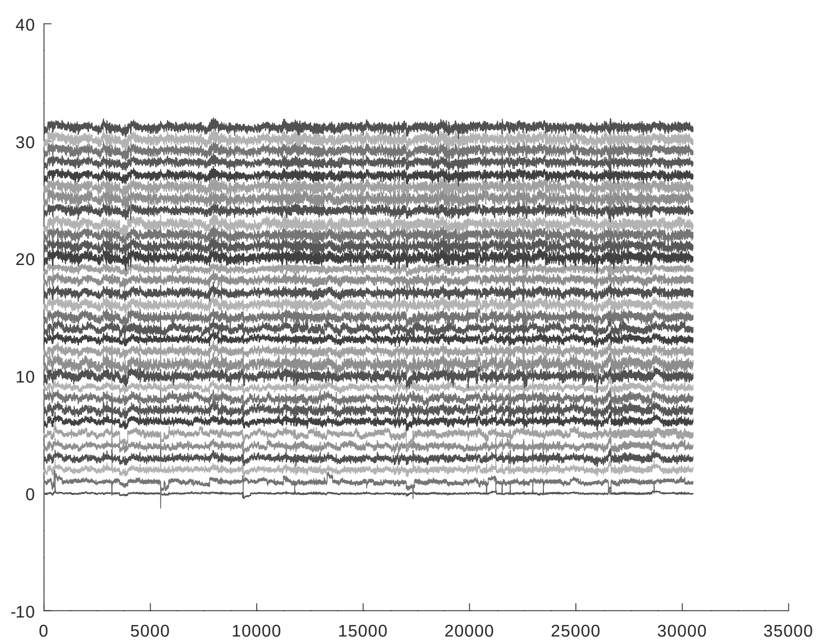

Figure 2.

1.3.1. Functional relationships in the data

Over some time subinterval can the data from a sensor be expressed as a function of data from another sensor?

```
octave] cd /home/student/courses/MATH547ML/homework/hw01;
octave] [m,n]=size(data);
octave] mt=5000:5005; plot(data(mt,1),data(mt,2),'-o'); print -deps hw01Fig02a.eps
octave] mt=6000:6005; plot(data(mt,1),data(mt,2),'-o'); print -deps hw01Fig02b.eps
octave] mt=7000:7005; plot(data(mt,1),data(mt,2),'-o'); print -deps hw01Fig02c.eps
octave]
```

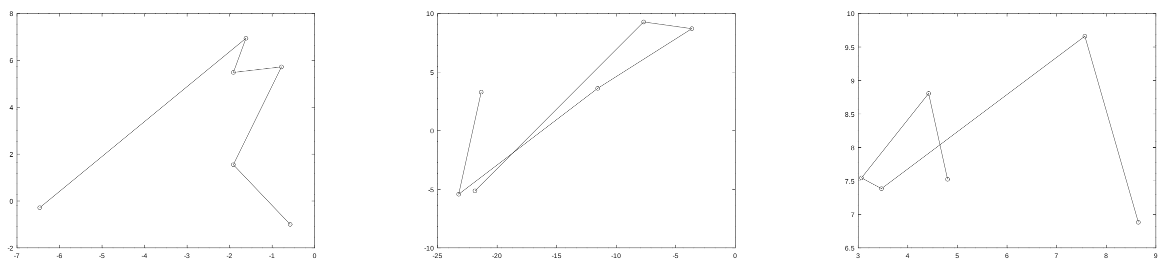

Figure 3. From above figures, sensors 1,2 are not in a functional relation.

1.3.2. Data magnitude (norm)

Partition sensor data into time subintervals. Over each subinterval, evaluate the p-norm of centered data, for $p = 1, 2, 3$, $p \to \infty$. Plot the p-norms over the entire time history.

```
octave] mud=mean(data);
octave] cdata=data-mud;
octave] m1=1000; m2=1100; cdata(m1:m2,:)
ans =

   30504       32

octave] function nrmd=normd(m1,m2,d)
          [m,n]=size(d);
          nrmd=zeros(4,n);
          for k=1:n
            nrmd(1,k)=norm(d(m1:m2,k),1);
            nrmd(2,k)=norm(d(m1:m2,k),2);
            nrmd(3,k)=norm(d(m1:m2,k),3);
            nrmd(4,k)=norm(d(m1:m2,k),Inf);
          end;
        end;
octave] tnrm=zeros(100,4,n);
        for j=1:100
          tnrm(j,:,:) = normd(100*j,100*(j+1),cdata);
        end;
octave] it=1:100; plot(it,tnrm(:,1,1),'r',it,tnrm(:,2,1),'k',it,tnrm(:,3,1),'g',it,
        tnrm(:,4,1),'b')
```

```
octave] plot(it,tnrm(:,1,1),'r',it,tnrm(:,1,2),'k',it,tnrm(:,1,3),'g')
```

```
octave] figure(); plot(it,tnrm(:,2,1),'r',it,tnrm(:,2,2),'k',it,tnrm(:,2,3),'g')
```

```
octave] figure();
```

```
octave] jt=8900:9200; plot(jt,cdata(jt,1),'r',jt,cdata(jt,2),'k',jt,cdata(jt,3),'g')
```

```
octave] figure(1); pwd
```

```
ans = /home/student/courses/MATH547ML/homework/hw01
```

```
octave] print -deps hw01fig03.eps
```

```
octave]
```

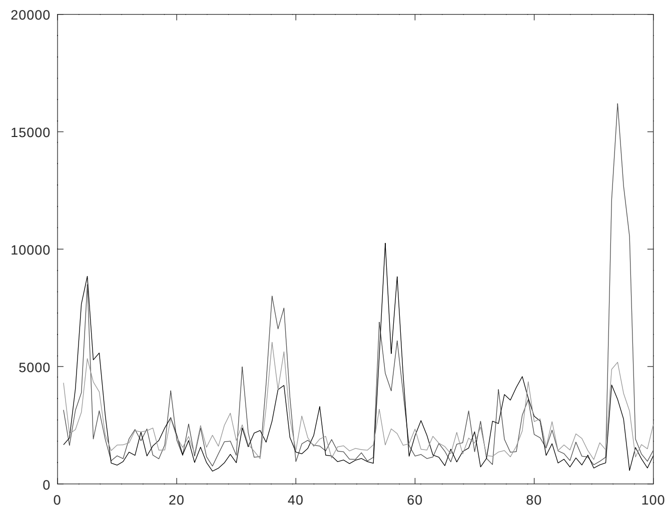

Figure 4.

1.3.3. Orthogonality of sensor data (inner product)

Over some time interval, determine the angle between sensor data. Is sensor data orthogonal? Plot the angle between sensor data over the entire time history.

```
octave] function angd=angld(m1,m2,n1,d)
          [m,n]=size(d);
          angd=zeros(n1,n);
          for k=1:n
            angd(k)=d(m1:m2,n1)'*d(m1:m2,k)/norm(d(m1:m2,n1))/norm(d(m1:m2,k));
          end
        end;
```

```
octave] tang=zeros(100,n);
        for j=1:100
          tang(j,:) = angld(100*j,100*(j+1),1,cdata);
        end;
```

```
octave] it=1:100; plot(it,tang(:,1),'r',it,tang(:,2),'k',it,tang(:,3),'g')
```

```
octave] figure(4)
```

```
octave] plot(tang)
```

```
octave]
```

1.3.4. Periodic representation of data (linear mapping)

Use template from Lesson01 to construct a representation of the data through periodic functions.

```
octave] ns=5; A=[]; it=100:400; dt=0.1; t=it*dt; t=t'; A = [A ones(size(t))];
octave] for k=1:ns
          A = [A sin(k*t) cos(k*t)];
        end
octave] bt=cdata(it,1);
octave] x=A\bt;
octave] b=A*x;
octave] plot(t,bt,'ok',t,b,'r');
octave] print -deps hw01fig04.eps
octave]
```

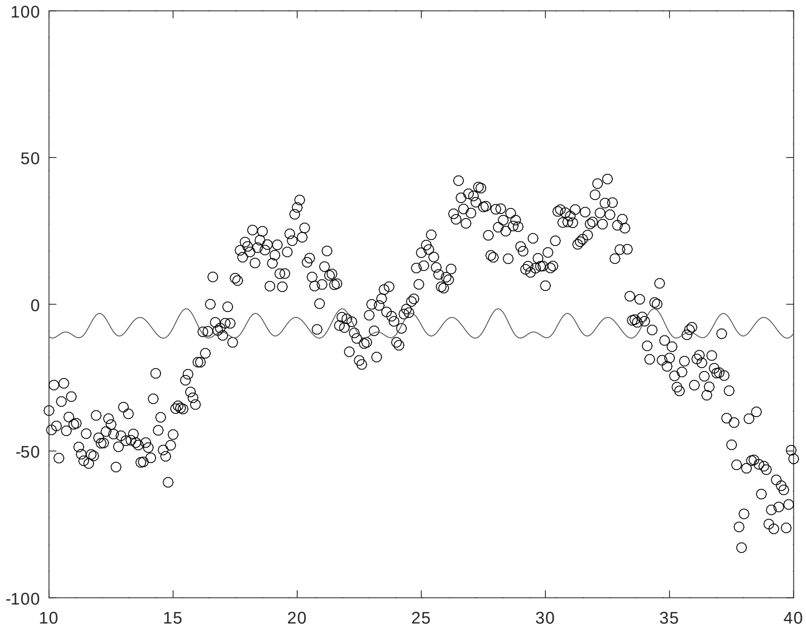

Figure 5. Data from sensor 1 is not accurately captured by a trigonometric series.