MATH 564: Final Examination

As a fitting capstone to this introduction to mathematical modeling in biology the exam is inspired by a current research problem, the propagation and focalization of shear waves in the brain as a possible mechanism for concussions. In this examination you will look at some of the simplest models for this problem.

As usual, your examination solution will be submitted through Sakai. You can choose to work in TeXmacs or directly in Mathematica. In both cases you will have to explain the approach you are taking, so it might be preferable to use TeXmacs.

First steps. Save this file as LastnameFinal.tm (or .nb if you choose to work in Mathematica) for submission to Sakai. The questions below should take no more than 90 minutes to solve. To allow for any unforseen difficulties with the electronic submission procedure, the time limit for submitting your examination solution is set to 11:55PM, Monday May 7, 2018, but you should try to submit within the allotted 3 hour examination time period 12:00-3:00PM. You are free to use any course materials in constructing your solution. Solutions must reflect your individual thinking, so do not discuss any aspect with anyone else. Have fun carrying out mathematical modeling of a real-life problem!

- Set working directory

1 Background

In a collaboration I have with investigators Gianmarco Pinto and Bharat Tripathi from the UNC Medical School, we are interested in the propagation of shear waves through the brain (Published paper describing model). The brain can be considered to be a soft, gel-like material, and the shear waves produce displacements of the tissue $u(x,t)$ (a length, measured in meters, with x spatial coordinate, t time) that satisfy the equation

$$\rho_0 \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x} + \frac{2\mu\beta}{3}\left(\frac{\partial u}{\partial x}\right)^3 \right) = 0, \tag{1}$$

with parameters: $\rho_0 = 10^3$ kg/m^3, $\mu = 10^3$ N/m^3, and $\beta = 1.5$ (nondimensional).

2 Questions

1. Classify the equation (1). What type of equation? Give all relevant classifications we have discussed in the course.

   Solution. Equation (1) is a nonlinear, partial differential equation of second order.

2. Is the equation (1) dimensionally homogeneous?

   Solution. Determine the units of each term in the equation, with $[A]$ denoting the units of quantity A

$$\begin{aligned}
\left[\rho_0 \frac{\partial^2 u}{\partial t^2}\right] &= [\rho_0]\frac{[u]}{[t^2]} = \frac{\text{kg}}{m^3}\frac{m}{s^2} = \frac{\text{kg}}{m^2\,s^2}\\
\left[\frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right)\right] &= \frac{1}{[x]}[\mu]\frac{[u]}{[x]} = \frac{1}{m}\frac{N}{m^3}\frac{m}{m} = \frac{1}{m^4}\frac{\text{kg}\,m}{s^2} = \frac{\text{kg}}{m^3\,s^2}\\
\left[\frac{\partial}{\partial x}\left(\frac{2\mu\beta}{3}\left(\frac{\partial u}{\partial x}\right)^3\right)\right] &= \frac{1}{[x]}[\mu]\frac{[u]^3}{[x]^3} = \frac{1}{m}\frac{N}{m^3}\frac{m^3}{m^3} = \frac{1}{m^4}\frac{\text{kg}\,m}{s^2} = \frac{\text{kg}}{m^3\,s^2}
\end{aligned}$$

It is apparent that, as stated, (1) is not dimensionally homogeneous. An equation cannot be valid if it is not dimensionally homogeneous, so there must be a mistake somewhere in the formulation (a common feature in the first stages of mathematical modeling!). Since both u and x are lengths, and kg/m^3 are correct units for mass density, backtrack what the units of μ should be

$$\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)\right]=\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)\right]\Rightarrow\frac{1}{[x]}[\mu]\frac{[u]}{[x]}=[\rho_0]\frac{[u]}{[t^2]}\Rightarrow[\mu]=[\rho_0]\frac{[u][x]}{[t^2]}=\frac{\text{kg}}{m^3}\frac{m^2}{s^2}=\frac{N}{m^2}.$$

Assume that $\mu=10^3\,N/m^2=10^3\,\text{Pa}$ henceforth.

3. Consider $\beta=0$. What changes in the classification of equation (1)? Does the problem become simpler for $\beta=0$? Why, and in what way?

   Solution. The equation remains a partial differential equation of second order, but is now linear. The problem becomes simpler since we can readily obtain simple analytical solutions for linear equations.

4. For $\beta=0$, verify that $u(x,t)=a\,f(x-ct)+b\,g(x+ct)$ is a solution of (1), with a, b arbitrary constants. Determine the expression for c in terms of the problem parameters.

   Solution. Start from dimensional homogeneity of $x-ct$ to deduce that $[c]=[x]/[t]=m/s$, hence c is a velocity. Now compute partial derivatives in (1), repeatedly using rule for differentiation of composite functions

$$\frac{\partial u}{\partial t}=-acf'+bcg',\ \frac{\partial^2 u}{\partial t^2}=ac^2f''+bc^2g'',$$

$$\frac{\partial u}{\partial x}=-af'+bg',\ \frac{\partial^2 u}{\partial x^2}=af''+bg''.$$

   Replace in (1) with $c=0$ to obtain

$$\rho_0c^2(af''+bg'')=\mu(af''+bg''),$$

   and deduce that, indeed, $u(x,t)=a\,f(x-ct)+b\,g(x+ct)$ is a solution if

$$c^2=\frac{\mu}{\rho_0}.$$

   Verify this conclusion by dimensional homogeneity:

$$[c^2]=\frac{m^2}{s^2}=\frac{[\mu]}{[\rho_0]}=\frac{Nm^3}{m^2\,\text{kg}}=\frac{m^2}{s^2},$$

   hence indeed c is a velocity, with value $c=1$ m/s for problem data

5. Let $\xi=x-ct$, and consider that $u(x,t)=a\,f(\xi)$. Rewrite (1) as an ordinary differential equation for $f(\xi)$, and find its solutions. Interpret the significance of each solution.

Solution. From composite differentiation rule, and with $c^2 = \mu / \rho_0$, obtain

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = a\,\mu\, f'', \mu \frac{\partial^2 u}{\partial x^2} = a \mu f'',$$

$$\frac{\partial}{\partial x}\left(\frac{2\mu\beta}{3} \left(\frac{\partial u}{\partial x} \right)^3 \right) = \mu \frac{\partial}{\partial x}\left(\left(\frac{\partial u}{\partial x} \right)^3 \right) = \mu \frac{\partial}{\partial x}(a^3 (f')^3) = \mu a^3\, 3\, (f')^2 f''.$$

Gathering terms, the PDE (1) reduces to an ODE

$$(f')^2 f'' = 0 \Rightarrow f' = 0 \text{ or } f'' = 0$$

with solutions

$$f_1(\xi) = C,\, f_2(\xi) = A\xi + B.$$

The first solution states that u is constant along lines of constant values of $\xi = x - ct$. The second solution states that u varies linearly as

$$u(x, t) = a(A\,(x - \mathrm{ct}) + B).$$

6. Now consider the ODE from (5., above) and add on a forcing term $\sin(x)$. Find numerical solutions to the resulting inhomogeneous ODE, and interpret the significance of the solution.

   Solution. The equation from (5., above) becomes

$$(f')^2 f'' = \sin(x) = \sin(\xi - ct). \tag{2}$$

   In this formulation ξ is the independent variable, f is the dependent variable, and t is a parameter, hence (2) is a family of second-order, nonlinear ODEs parametrized by t (time). To solve the problem, two boundary conditions must be imposed (second-order differential equation). Recall that $u(x,t) = a\,f(x - ct) = a\,f(\xi)$ is a displacement. An interesting set of boundary conditions would be

$$f(0) = \sin(t),\, f'(0) = \cos(t)$$

   This corresponds to wiggling the end at $\xi = 0$, with displacement $\sin(t)$ and velocity $\cos(t)$. Here's a straightforward implementation of the above ideas.

```
In[11]:= eq[t_]:=(f'[csi])^2 f''[csi] == Sin[csi-t];
         cond[t_]:={f[0]==Sin[t], f'[0]==Cos[t]};
         sol[t_]:= NDSolve[ Flatten[{eq[t],cond[t]}], f, {csi,0,
         1}][[1,1]];
```

   Null

```
In[12]:=
```

   Test the above implementation to find the solution at $t = 0$

```
In[17]:= p0=Plot[Evaluate[f[csi] /. sol[0.]],{csi,0,1},Frame->True,
         FrameLabel->{"csi","u[csi]"},GridLines->Automatic];
         Export["p0.png",p0];
```

   Null

```
In[18]:=
```

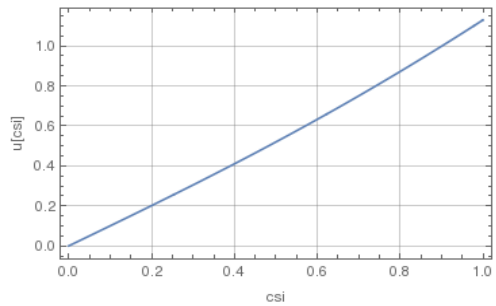

Figure 1. Solution at $t=0$

Note: the above directly uses the syntax from the Mathematica NDSolve example. And now generate a number of solutions at various values of the parameter t

```
In[30]:= plots=Table[ Plot[Evaluate[f[csi] /. sol[t]],{csi,0,
         1},Frame->True,FrameLabel->{"csi","u[csi]"},GridLines-
         >Automatic],{t,0.,.5,0.05} ];
         Export["plots.png",Show[plots]];
In[31]:=
```

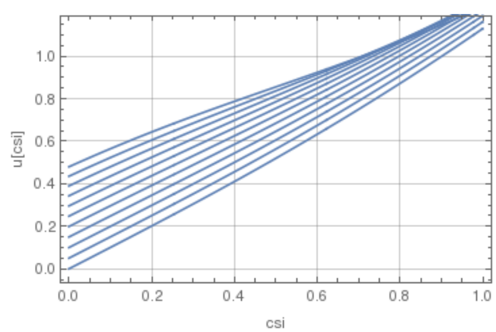

Figure 2. Solutions at $t=0, 0.05, ..., 0.5$

7. Change the forcing term to $\sin(kx)$, and investigate the effect on the numerical solution of varying k.

   Solution. This is similar to above formulation

```
In[33]:= eq[t_,k_]:=(f'[csi])^2 f''[csi] == Sin[k(csi-t)];
         cond[t_]:={f[0]==Sin[t], f'[0]==Cos[t]};
         sol[t_,k_]:= NDSolve[ Flatten[{eq[t,k],cond[t]}], f, {csi,0,
         1}][[1,1]];
```

Null

```
In[35]:= kplots=Table[ Plot[Evaluate[f[csi] /. sol[0.,k]],{csi,
         0,1},Frame->True,FrameLabel->{"csi","u[csi]"},GridLines-
         >Automatic],{k,0.,100.,10.} ];
         Export["kplots.png",Show[kplots]];
```

Null

In[36]:=

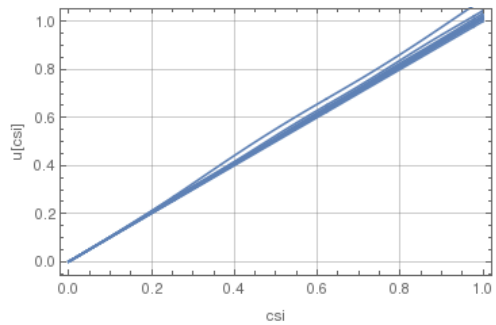

Figure 3. Solutions at $t = 0$, for $k = 0, 10, \ldots, 100$