Discrete two-dimensional model of infection

1 Problem formulation

Consider $S(t, x, y), I(t, x, y)$ represent the suceptible, infectious populations as functions of time t, and position in space (x, y). The evolution of these populations follows the model

$$\frac{\partial S}{\partial t} = -\beta S\,(I + \alpha\nabla^2 I) + \delta\nabla^2 S$$
$$\frac{\partial I}{\partial t} = \beta S\,(I + \alpha\nabla^2 I) - \gamma I$$
$$S, I\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$$

The above continuous model can be reduced to a discrete representation by:

1. Finite difference approximation on regular grids

2. Graph approximation on irregular grids

1.1 Regular grid

Consider $x_i = ih,\ y_j = jh$ to be positions in a Cartesiantwo-dimensional grid, with h the step size. Also introduce $t^n = nk$ to represent a sequence of discrete moments in time, with k the time step. Let S_{ij}^n denote the value of $S(t^n, x_i, y_j)$. The time derivative can be approximated as

$$\frac{\partial S}{\partial t}(t^n, x_i, y_j) \cong \frac{S_{ij}^{n+1} - S_{ij}^n}{k}$$

The spatial derivatives in the ∇^2 operator can similarly be approximated

$$\nabla^2 S(t^n, x_i, y_j) = \frac{\partial^2 S}{\partial x^2}(t^n, x_i, y_j) + \frac{\partial^2 S}{\partial y^2}(t^n, x_i, y_j) \cong \frac{1}{h^2}(S_{i+1,j}^n + S_{i-1,j}^n + S_{i,j+1}^n + S_{i,j-1}^n - S_{\text{ij}}^n).$$

From above approximations the discrete evolution laws that correspond to the SIR model are

$$\frac{S_{ij}^{n+1} - S_{ij}^n}{k} = -\beta S_{ij}^n\Big[I_{ij}^n + \frac{\alpha}{h^2}(I_{i+1,j}^n + I_{i-1,j}^n + I_{i,j+1}^n + I_{i,j-1}^n - I_{\text{ij}}^n)\Big] + \frac{\delta}{h^2}(S_{i+1,j}^n + S_{i-1,j}^n + S_{i,j+1}^n + S_{i,j-1}^n - S_{\text{ij}}^n)$$

$$\frac{I_{ij}^{n+1} - I_{ij}^n}{k} = \beta S_{ij}^n\Big[I_{ij}^n + \frac{\alpha}{h^2}(I_{i+1,j}^n + I_{i-1,j}^n + I_{i,j+1}^n + I_{i,j-1}^n - I_{\text{ij}}^n)\Big] - \frac{\gamma}{h^2} I_{ij}^n$$

The discrete update rules are

$$S_{ij}^{n+1} = S_{ij}^n - a S_{ij}^n I_{ij}^n - b S_{ij}^n\,\nabla_h^2 I_{\ ij} + c\nabla_h^2 S_{\ ij}$$

$$I_{ij}^{n+1} = I_{ij}^n + a S_{ij}^n I_{ij}^n + b S_{ij}^n\,\nabla_h^2 I_{\ ij} - d I_{\ ij}$$

with notation

$$\nabla_h^2 S_{\ ij} = \frac{1}{h^2}(S_{i+1,j}^n + S_{i-1,j}^n + S_{i,j+1}^n + S_{i,j-1}^n - S_{\text{ij}}^n)$$