Course overview:

- Models in the life sciences
- Software to formulate and solve models

Review of some mathematical tools:

- Linear dependence

- Living entities exhibit:
  - Spatial organization, they are separated from their environment
  - Metabolism, they undergo internal physico-chemical processes
  - Homeostasis, they maintain a quasi-stable internal state
  - Reproduction, they produce copies of themselves, perhaps mutated
  - Adaptation, favorable mutations multiply more rapidly, they evolve
  - Stimuli response, internal changes occur upon environment modification
- All the above exhibit regularity, and are therefore the object of scientific study
- Mathematical biology seeks development of well-formulated models to answer biological questions

- Biology identifies populations of distinct species that interact, e.g., predation
- Biological question: what are stable populations of predators/prey?
- Mathematical formulation: hypotheses

  a. population numbers $y(t)$ (prey), $z(t)$ (predator) are functions
  b. populations are large $y, z \gg 1$
  c. difference $y - [y] \in [0, 1)$ is negligible ($[y]$ is the integer part of y)
  d. (a)-(c) imply that y, z are continuous
  e. time scale of population change is much larger than predation time interval
  f. many possible predation encounters
  g. (e)-(f) imply that $y(t), z(t)$ differentiable
  h. prey has positive natural growth rate a, predation decrease probability p
  i. predators have a negative growth rate $-b$, feeding increase probability q

- Hypotheses (a)-(i) lead to a system of first-order differential equations

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= ay - pyz \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= -bz + qyz\end{aligned}$$

- As in many biological settings, the above system is nonlinear and typically does not admit an analytical solution so numerical approaches are required

- Though analytical solutions are either impossible or tedious to obtain, numerical and approximate solutions are possible
- Modern computational, symbolic software systems easily construct solutions
- Lotka-Volterra system formulation in Mathematica

```
In[66]:= PreyEq = y'[t] == a y[t] - p y[t] z[t];
         PredEq = z'[t] == -b y[t] + q y[t] z[t];
         LVparams = {a->0.7, b->1, p->1.3, q->1};
         ICs = {y[0]==1,z[0]==0.5};
         LVsystem =Flatten[{PreyEq,PredEq,ICs} /. LVparams]
```

- Numerical solution

```
In[79]:= sol = NDSolve[LVsystem,{y[t],z[t]},{t,0,1.5}][[1]]
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$$\{y'(t) = 0.7\,y(t) - 1.3\,y(t)\,z(t), z'(t) = y(t)\,z(t) - y(t), y(0) = 1, z(0) = 0.5\}$$

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```

$$\{y(t) \to \text{InterpolatingFunction}[](t), z(t) \to \text{InterpolatingFunction}[](t)\}$$

- Graphical presentation

```
In[80]:= Plot[Evaluate[{y[t],z[t]} /. sol], {t,0,1.5},PlotLegends->{"Prey",
         "Pred"}]
In[81]:=
```

- Graphical presentation

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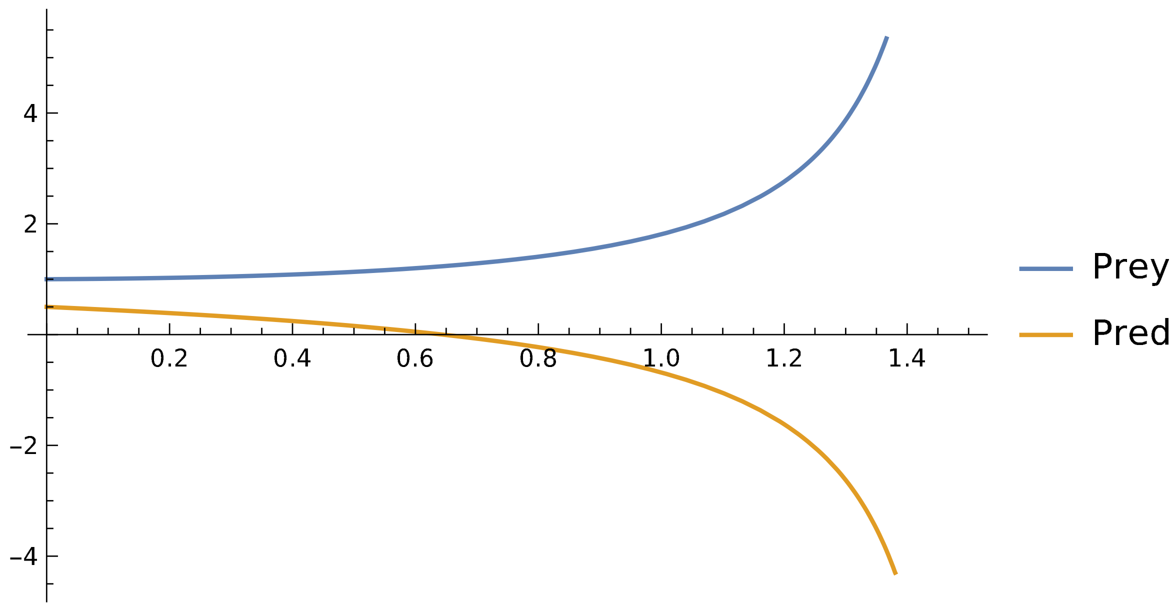
```




```
In[81]:=
```

- $\boldsymbol{x} \in \mathbb{R}^n$ independent, $\boldsymbol{y} \in \mathbb{R}^m$ dependent variables, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ a matrix

$$\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} \tag{1}$$

  is a linear relationship that satisfies

$$\boldsymbol{y}(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2) = \alpha_1 \boldsymbol{y}(\boldsymbol{x}_1) + \alpha_2 \boldsymbol{y}(\boldsymbol{x}_2)$$

$$\boldsymbol{A}(\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2) = \alpha_1 \boldsymbol{A}\boldsymbol{x}_1 + \alpha_2 \boldsymbol{A}\boldsymbol{x}_2$$

  for any scalars $\alpha_1, \alpha_2 \in \mathbb{R}$

- For $m = n = 1$, scalar linear relation $y = kx$, e.g., $y = \frac{9}{5}x$, with $y = F - 32$ the Fahrenheit temperature above the melting point of ice at standard atmospheric pressure and $x = C$ the Celsius temperature.

```
In[84]:= SetDirectory[$HomeDirectory <> "/courses/MATH564/lessons"];
         p = Plot[ 9c/5+32, {c,0,100},
                   Frame->True,FrameLabel->{"Temp C","Temp F"},
                   GridLines->Automatic];
         Export["CelsiusToFahrenheit.png",p]
```

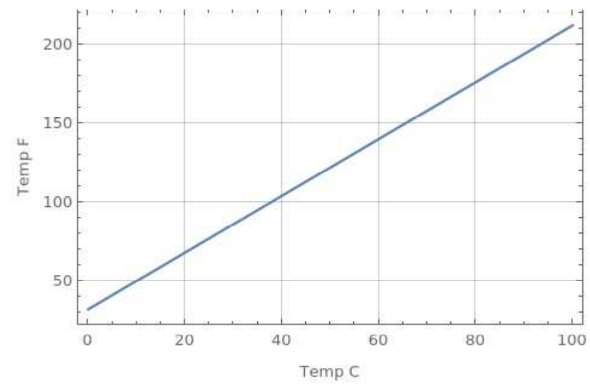


Figure 1. Linear dependence plot

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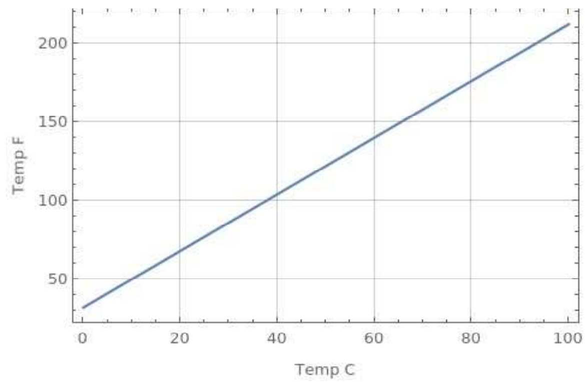

Figure 2. Linear dependence plot

- Transform $y = Ax^k$ into linear dependence of $\log y$ on $\log x$,

$$\log y = k \log x + \log A$$

```
In[87]:= y = A x^k
In[88]:= PowerExpand[Log[y]]
In[89]:= {Expand[(a+b)^3],TrigExpand[Cos[a+b]]}
In[90]:=
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```

$\{a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3, \cos(a)\cos(b) - \sin(a)\sin(b)\}$

```
In[90]:=
```