Review of some mathematical tools:

- Differential equations
- Difference equations

- The ordinary differential equations (ODEs) of order k

$$\boldsymbol{y}^{(k)} = \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{y}', \ldots, \boldsymbol{y}^{(k-1)})$$

  expresses the change in the m dependent variables $\boldsymbol{y}(t) \in \mathbb{R}^m$ through growth rates $\boldsymbol{f}\colon \mathbb{R} \times \mathbb{R}^{m \times k} \to \mathbb{R}^m$

- A scalar ODE is obtained for $m = 1$

$$y^{(k)} = f(t, y, \ldots, y^{(k-1)})$$

  with $f\colon \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$

- A first-order scalar ODE is obtained for $m = 1$, $k = 1$

$$y' = f(t, y)$$

- Any ODEs of order $k > 1$ can be transformed into a system of first-order ODEs

$$\boldsymbol{y}^{(k)} = \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{y}', \ldots, \boldsymbol{y}^{(k-1)})$$
$$\text{Introduce } \boldsymbol{z}_1 = \boldsymbol{y}, \boldsymbol{z}_2 = \boldsymbol{y}', \ldots, \boldsymbol{z}_k = \boldsymbol{y}^{(k-1)}$$
$$\boldsymbol{z} = \begin{pmatrix} \boldsymbol{z}_1 \\ \vdots \\ \boldsymbol{z}_k \end{pmatrix} \in \mathbb{R}^{km}$$
$$\boldsymbol{z}' = \begin{pmatrix} \boldsymbol{z}_1' \\ \vdots \\ \boldsymbol{z}_k' \end{pmatrix} = \boldsymbol{F}(t, \boldsymbol{z}) = \begin{pmatrix} \boldsymbol{z}_2 \\ \vdots \\ \boldsymbol{f}(t, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_k) \end{pmatrix}$$

- A first-order system $\boldsymbol{z}' = \boldsymbol{F}(t, \boldsymbol{z})$ is linear if $\boldsymbol{F}(t, \boldsymbol{z}) = \boldsymbol{A}(t)\,\boldsymbol{z}$

$$\boldsymbol{z}' = \boldsymbol{A}(t)\,\boldsymbol{z},$$

with $\boldsymbol{A} \in \mathbb{R}^{km \times km}$ a matrix.

- Linear ODE of first order $y' = ky$ (Eq. 2.4.1, p.37), with initial condition $y'(0) = 1$

```
In[2]:= DSolve[y'[t]==k y[t],y[t],t]
```

```
In[4]:= DSolve[{y'[t]==k y[t],y[0]==1},y[t],t][[1,1]]
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- Nonlinear, first-order ODE $y' = y - y^2/(2+\sin t)$ (Eq. 2.4.3, p. 37)

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In[5]:= DSolve[y'[t]==y[t]-y[t]^2/(2+Sin[t]),y[t],t][[1,1]]
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$$y(t) \to \frac{e^t}{-\int_1^t -\frac{e^{K[1]}}{\sin(K[1]) + 2} \, dK[1] + c_1}$$

- Linear, second-order ODE $y'' - 4y' + 4y = e^{-t}$ (Eq. 2.4.2, p.37)

```
In[7]:= DSolve[{y''[t]-4 y'[t]+4 y[t]==Exp[-t]},y[t],t][[1,1]]
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- And with initial conditions, $y(0) = 0$, $y'(0) = 1$

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In[8]:= DSolve[{y''[t]-4 y'[t]+4 y[t]==Exp[-t],y[0]==0,y'[0]==1},y[t],t][[1,1]]
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$$y(t) \to \frac{e^{-t}}{9} + c_1\, e^{2t} + c_2\, e^{2t}\, t$$

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```

$$y(t) \to \frac{1}{9}\, e^{-t}\,(12\, e^{3t}\, t - e^{3t} + 1)$$

```
In[9]:=
```

- Some ODEs cannot be easily integrated analytically (or at all). In such cases a numerical solution can be constructed (see Homework01.nb)

```
In[4]:= sol = NDSolve[{y'[t]==y[t]-y[t]^2/(2+Sin[t]),y[0]==1},y[t],{t,0,1}][[1,
        1]];
        numsol[t_] := y[t] /. sol;
```

```
In[6]:= Plot[numsol[t],{t,0,1},PlotLabel->"Numerical ODE solution",GridLines-
        >Automatic,Frame->True,FrameLabel->{"t","y(t)"}]
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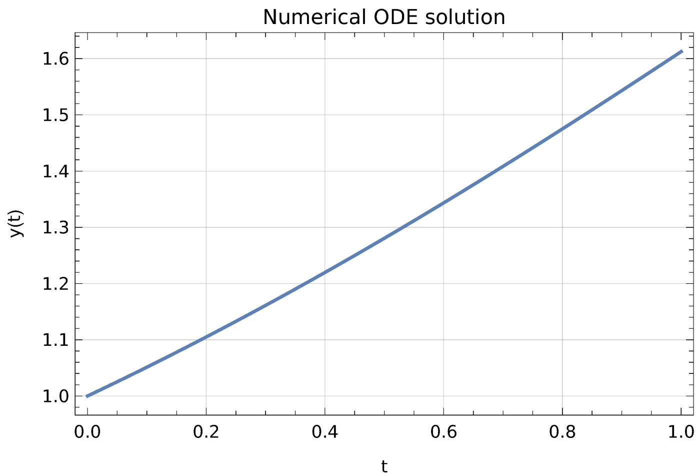

- Newton's second law often leads to second-order ODE systems for which insight is often obtained from plots of the dependent variable and its derivative

```
In[1]:= sol[t_] =
         y[t] /. DSolve[ {y''[t] + y'[t] + 2 y[t] == 0, y[0] == 1,
             y'[0] == 0}, y[t], t][[1, 1]]
```

```
In[2]:= ParametricPlot[{sol[t], sol'[t]}, {t, 0, 20}, GridLines -> Automatic,
        Frame -> True, FrameLabel -> {"y", "y'"},
         PlotLabel -> "Phase portrait", PlotRange -> All]
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- In the above, the system decays to zero.

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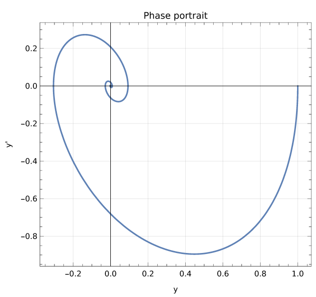

- In the above, the system decays to zero.

- Linear forced systems often lock on to the forcing

```
In[3]:= DEq = y''[t] + y'[t] + 2 y[t] == Sin[t];
        InC = { y[0] == 1, y'[0] == 0};
        IVP = Flatten[{DEq, InC}];
        sol[t_] = y[t] /. DSolve[ IVP, y[t], t][[1, 1]];
        ParametricPlot[{sol[t], sol'[t]}, {t, 0, 20}, GridLines -> Automatic,
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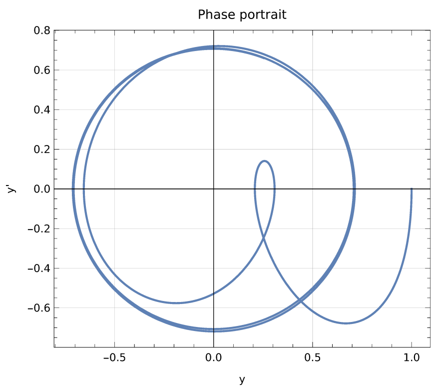

- Nonlinear forced systems often exhibit complex dynamics

```
In[8]:= DEq = y''[t] + Sin[ y[t]] == Sin[t];
        InC = { y[0] == 1, y'[0] == 1};
        IVP = Flatten[{DEq, InC}];
        tf = 200;
        sol[t_] = y[t] /. NDSolve[IVP, y[t], {t, 0, tf}][[1, 1]];
        Plot[{sol[t], sol'[t]}, {t, 0, tf}]
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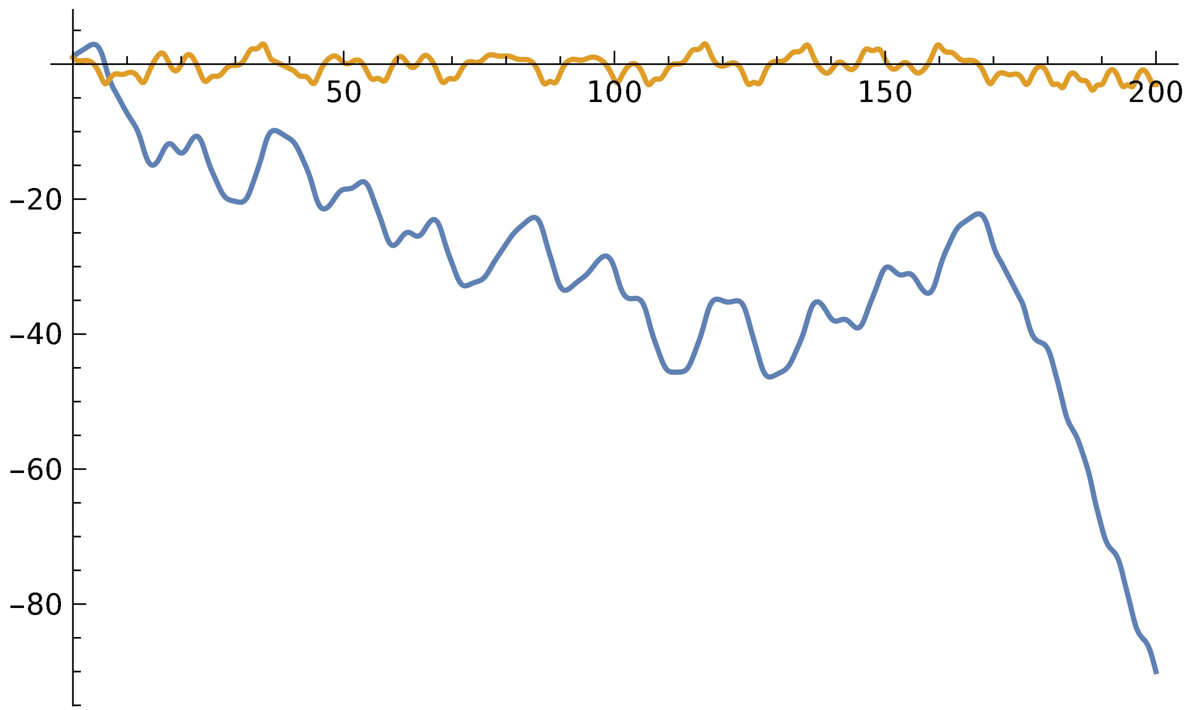

- Phase portrait for system from previous slide

```
In[14]:= phsplt =
    ParametricPlot[{sol[t], sol'[t]}, {t, 0, tf}, GridLines -> Automatic,
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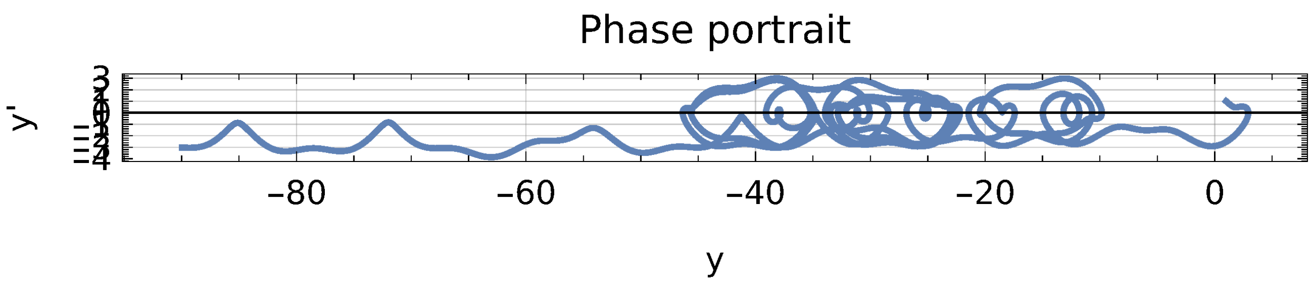

- ODE $y' = f(t, y)$ describes rates of change over a continuum of times
- Similarly, the equation $y_{n+1} - y_n = f(y_n)$ is a first-order difference equation to describe events occuring at distinct times
- Homogeneous, linear difference equation of order k

$$y_{n+k} + c_{k-1}\, y_{n+k-1} + c_{k-2} y_{n+k-2} + \cdots + c_0 y_n = 0$$

- Guess solutions of form $y_n = A r^n$ to obtain characteristic equation

$$r^k + c_{k-1} r^{k-1} + \cdots + c_1 r + c_0 = 0$$

  a polynomial with k roots $r_1, \ldots, r_k$
- Solution is

$$y_n = c_1 r_1^n + \cdots + c_k r_k^n$$

  with constants $c_1, \ldots, c_k$ determined from initial conditions $y_0, \ldots, y_{k-1}$.

- A diverging system $y_{n+1} - y_n = (-1)^n n,\ y_0 = 1$

```
In[15]:= sol=RSolve[y[n+1]-y[n]==(-1)^n n,y[n],n][[1,1]]
```

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In[1]:= sol1=RSolve[{y[n+1]-y[n]==(-1)^n n, y[0]==1},y[n],n][[1,1]]
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In[3]:= ListPlot[ Table[{n,Evaluate[y[n] /. sol1]},{n,0,10}], Frame->True,
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$$y(n) \to \frac{1}{4}\left(-2\,(-1)^n\, n + (-1)^n - 1\right) + c_1$$

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