- Models for controlled population growth (r, K strategies)
- Community ecology
- Environmental limits on population growth

- Robert Burns (1759-1796), *To a mouse* (1785)

But, Mousie, thou art no thy-lane,
In proving foresight may be vain;
The best-laid schemes o' mice an' men
Gang aft agley,
An' lea'e us nought but grief an' pain,
For promis'd joy!

Still thou art blest, compar'd wi' me
The present only toucheth thee:
But, Och! I backward cast my e'e.
On prospects drear!
An' forward, tho' I canna see,
I guess an' fear!

- John Steinbeck (1937) George Milton and Lennie Small during the Great Depression
- R. H. MacArthur, E. O. Wilson, *The theory of island biogeography*, Princeton Univ. Press (1967), E. Pianka *The American Naturalist*, **104**(940):592-597, 1970: On r- and K- selection

$$y' = f(y)y = r\,(1 - y/K)\,y$$

- Solve $y' = ry \cdot (1 - y/K)$ by separation of variables

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ry \cdot (1 - y/K) \Rightarrow \int \frac{\mathrm{d}y}{y \cdot (1 - y/K)} = r \int \mathrm{d}t - \ln C = rt - \ln C$$

$$\int \frac{\mathrm{d}y}{y \cdot (1 - y/K)} = \int \left[\frac{1}{y} - \frac{1}{y - K}\right] \mathrm{d}y = \ln y - \ln(y - K) = \ln \frac{y}{y - K}$$

- Effect of changes in growth rate r, environmental carrying capacity K

```
In[11]:= Off[Solve::ifun];
         y[t_,r_,K_,z0_] = z[t] /. DSolve[{z'[t]==r z[t] (1-z[t]/K),z[0]==z0},z[t],
         t][[1,1]];
         logisticPlots=Plot[Table[y[t, r, K,0.5], {r, 0.25, 1, 0.25}, {K, 0.5, 1.5,
         0.5}], {t, 0, 10}, Frame -> True, FrameLabel -> {"time", "population"},
         PlotLabel -> "Logistic population model", GridLines -> Automatic];
         SetDirectory[$HomeDirectory <> "/courses/MATH564/lessons/images"];
         Export["logisticPlots.png",logisticPlots]
```

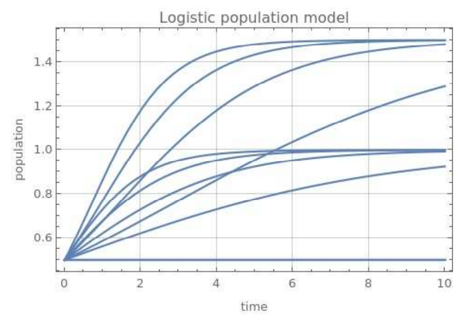

Figure 1. Logistic equation solutions for diverse r, K

- Effect of changes in growth rate r, environmental carrying capacity K

```
In[11]:= Off[Solve::ifun];
         y[t_,r_,K_,z0_] = z[t] /. DSolve[{z'[t]==r z[t] (1-z[t]/K),z[0]==z0},z[t],
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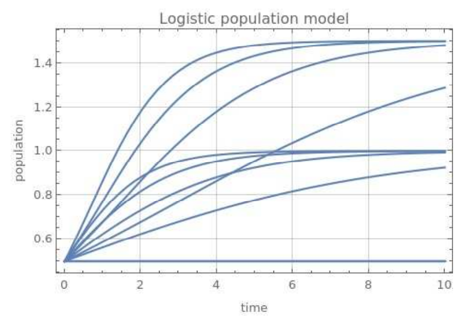

Figure 2. Logistic equation solutions for diverse r, K

- Logistic solution $y(t;r,K) = \dfrac{K\,e^{rt}\,y_0}{K + y_0(e^{rt}-1)}$, is nonlinear in parameters r, K

```
In[23]:= tau= 0.1; data = Table[{t, y[t, 0.5, 2]}, {t, 0, 10, tau}];
         Normal[NonlinearModelFit[data, y[t, r, K], {{r, 1}, {K, 10}}, t]]
```

- Even affected by measurement error, the nonlinear fit recovers parameters

```
In[25]:= data = data+Table[{0, 0.05 Random[]}, {t, 0, 10, tau}];
         Normal[NonlinearModelFit[data, y[t, r, K], {{r, 1}, {K, 10}}, t]]
```

- However, the nonlinear fitting procedure is not guaranteed to converge

```
In[28]:= Normal[NonlinearModelFit[data, y[t, r, K], {{r, 0.01}, {K, 10}}, t]]
```

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$$\frac{2.\,e^{0.5t}}{1.+e^{0.5t}}$$

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```

$$\frac{2.01816\,e^{0.522114t}}{1.01816+e^{0.522114t}}$$

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- However, the nonlinear fitting procedure is not guaranteed to converge

```
In[28]:= Normal[NonlinearModelFit[data, y[t, r, K], {{r, 0.01}, {K, 10}}, t]]
```

$$-\frac{1.58878\times 10^{15}\,e^{0.0890411t}}{e^{0.0890411t}-1.58878\times 10^{15}}$$

- Discrete equivalent of $y' = ry \cdot (1 - y/K)$ is $y_{i+1} - y_i = \rho y_i \cdot (1 - y_i/K)$
- Some solutions are close to those of the differential model, but others differ!

```
In[29]:= yd[rho_, K_, t1_]:=RecurrenceTable[ {z[t+1]-z[t] == rho z[t] (1-z[t]/K),
         z[0] == 1}, z[t], {t, 0, t1}];
```

```
In[30]:= logisticdifPlots=GraphicsGrid[Table[ListPlot[yd[r, k, 100], Frame->True,
         Joined->True, GridLines -> Automatic], {k, 2., 3., 1.}, {r, 1., 3.,
         1.}]];
         Export[$HomeDirectory <> "/courses/MATH564/lessons/images/
         logisticdifPlots.png",logisticdifPlots];
```

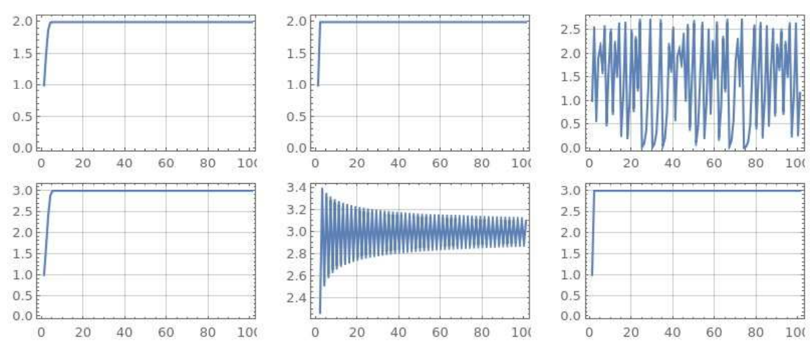

Figure 3. Behavior of discrete logistic model

- Discrete equivalent of $y' = ry \cdot (1 - y/K)$ is $y_{i+1} - y_i = \rho y_i \cdot (1 - y_i/K)$
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         z[0] == 1}, z[t], {t, 0, t1}];
```

```
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         Joined->True, GridLines -> Automatic], {k, 2., 3., 1.}, {r, 1., 3.,
         1.}]];
         Export[$HomeDirectory <> "/courses/MATH564/lessons/images/
         logisticdifPlots.png",logisticdifPlots];
```

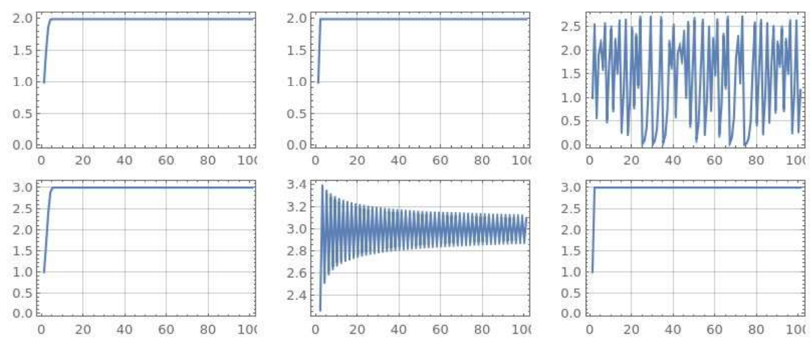

Figure 4. Behavior of discrete logistic model

- Discrete equivalent of $y' = ry \cdot (1 - y/K)$ is $y_{i+1} - y_i = \rho y_i \cdot (1 - y_i/K)$
- Some solutions are close to those of the differential model, but others differ!

```
In[29]:= yd[rho_, K_, t1_]:=RecurrenceTable[ {z[t+1]-z[t] == rho z[t] (1-z[t]/K),
         z[0] == 1}, z[t], {t, 0, t1}];
```

```
In[30]:= logisticdifPlots=GraphicsGrid[Table[ListPlot[yd[r, k, 100], Frame->True,
         Joined->True, GridLines -> Automatic], {k, 2., 3., 1.}, {r, 1., 3.,
         1.}]];
         Export[$HomeDirectory <> "/courses/MATH564/lessons/images/
         logisticdifPlots.png",logisticdifPlots];
```

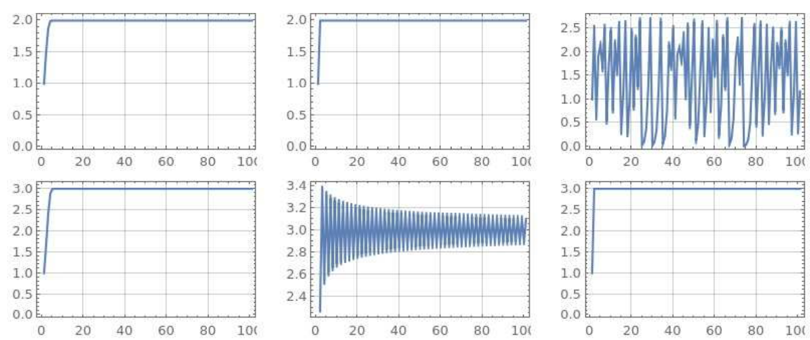

Figure 5. Behavior of discrete logistic model