- Lotka-Volterra model analysis
  - phase portraits
  - stability analysis
- Models with more than 2 species

- Species populations $x(t), y(t)$ change in time due to:
  - positive fertility r for prey, negative fertility $-m$ for predator
  - predation effect on prey $-axy$, on predator bxy

$$x' = rx - axy, y' = -my + bxy$$

- First-order, nonlinear ODE system that typically requires numerical solution

```
In[32]:= {r, a, m, b} = {2.5, 0.001, .2, 0.1}; (* model parameters *)
         PreyDE = x'[t] == r x[t] - a x[t] y[t];
         PredDE = y'[t] == -m y[t] + b x[t] y[t];
         PreyInit = x[0] == 1000;
         PredInit = y[0] == 20;
         LVmodel = {PreyDE, PredDE, PreyInit, PredInit};
```

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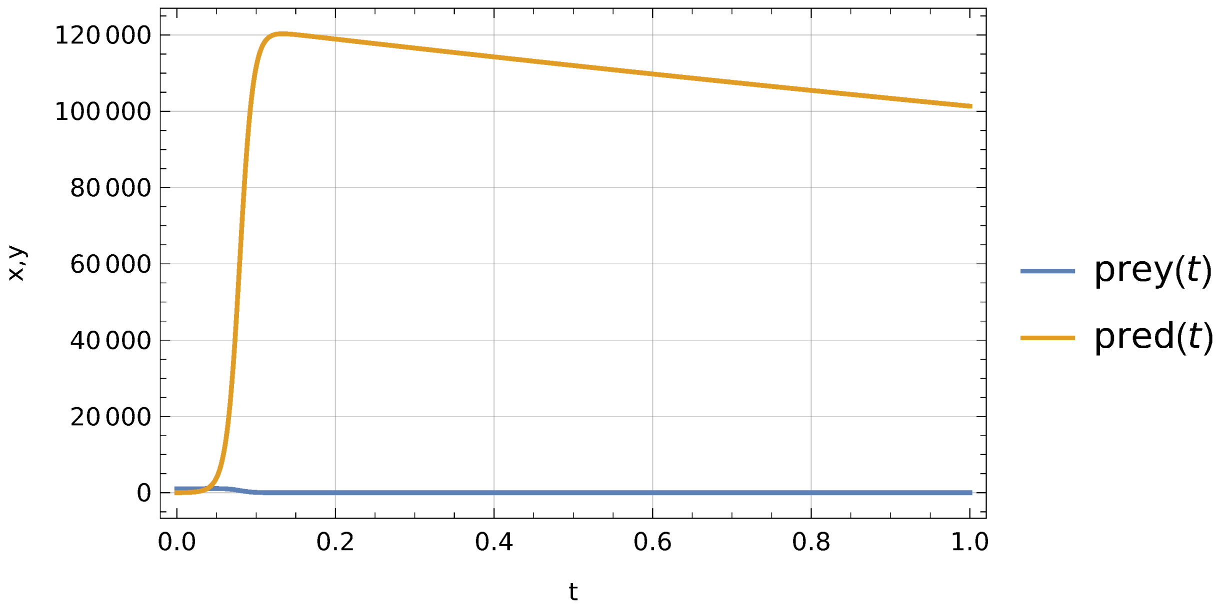
         PredInit = y[0] == 20;
         LVmodel = {PreyDE, PredDE, PreyInit, PredInit};
```

```
In[40]:= T = 100;
         sol = NDSolve[LVmodel, {x[t], y[t]}, {t, 0, 10}][[1]];
         prey[t_] = x[t] /. sol;
         pred[t_] = y[t] /. sol;
         Plot[{prey[t], pred[t]}, {t, 0, 1}, Frame -> True,
          GridLines -> Automatic, FrameLabel -> {"t", "x,y"},
          PlotLegends -> "Expressions"]
```

```
In[45]:=
```

```
In[40]:= T = 100;
         sol = NDSolve[LVmodel, {x[t], y[t]}, {t, 0, 10}][[1]];
         prey[t_] = x[t] /. sol;
         pred[t_] = y[t] /. sol;
         Plot[{prey[t], pred[t]}, {t, 0, 1}, Frame -> True,
          GridLines -> Automatic, FrameLabel -> {"t", "x,y"},
          PlotLegends -> "Expressions"]
```




```
In[45]:=
```

- Another presentation of solutions is to plot $y(x)$

```
In[46]:= ClearAll[m, b, r, a, x, y, x, y, sol];
         sol[x_, r_, a_, m_, b_, c_] = (y[x] /.
           DSolve[{y'[x] == (-m + b x)/(r - a y[x]) y[x]/x}, y[x],x][[1, 1]] ) /
         . C[1] -> c
         Plot[Table[sol[x, 2.5, 0.001, .2, 0.1, c], {c, 0, 10}], {x, 0, 20},
         Frame -> True, FrameLabel -> {"x", "y"}, GridLines -> Automatic]
```

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           DSolve[{y'[x] == (-m + b x)/(r - a y[x]) y[x]/x}, y[x],x][[1, 1]] ) /
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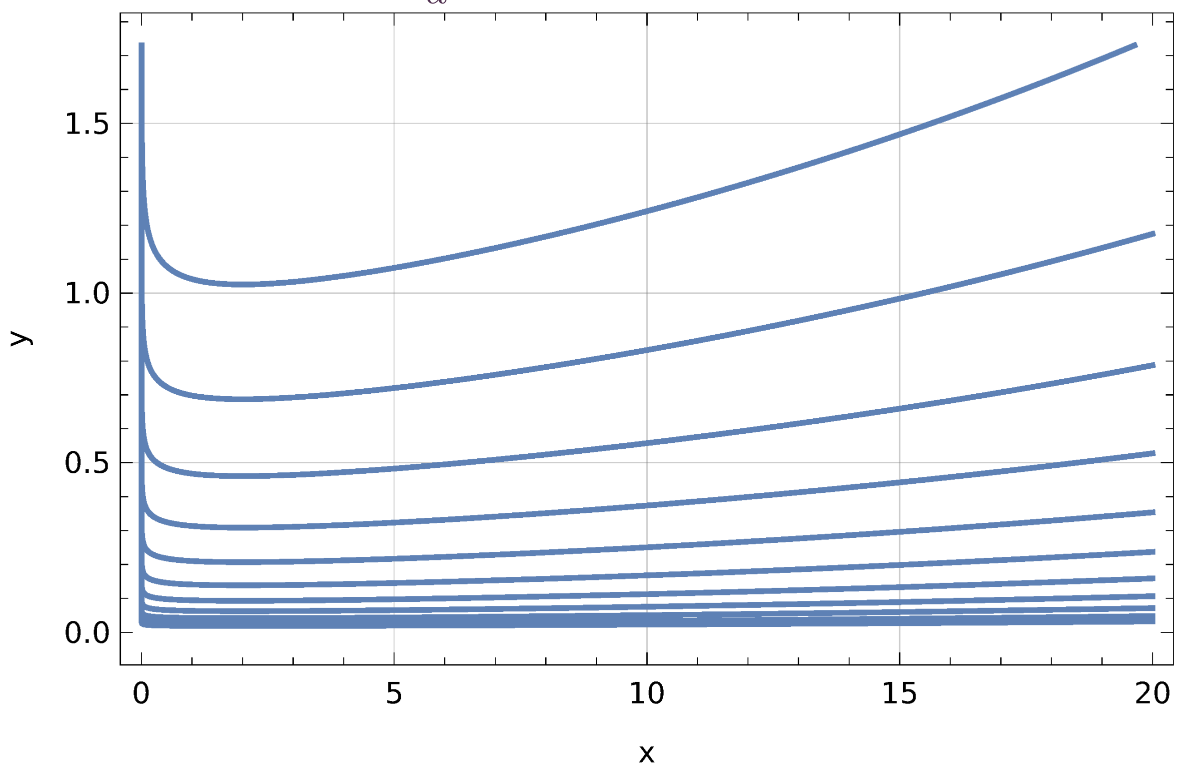
         Plot[Table[sol[x, 2.5, 0.001, .2, 0.1, c], {c, 0, 10}], {x, 0, 20},
         Frame -> True, FrameLabel -> {"x", "y"}, GridLines -> Automatic]
```

Out[47]= $-\frac{r\, W\left(-\frac{a\, x^{-\frac{m}{r}}\, e^{\frac{b x}{r}-\frac{c}{r}}}{r}\right)}{a}$



- More informative presentation: construct the parametric curve $(x(t), y(t))$

```
In[49]:= {r, a, m, b} = {1, 1, 1, 1}; (* model parameters *)
         PreyDE = x'[t] == r x[t] - a x[t] y[t];
         PredDE = y'[t] == -m y[t] + b x[t] y[t];
         PreyInit = x[0] == 1.5;
         PredInit = y[0] == 0.5;
         LVmodel = {PreyDE, PredDE, PreyInit, PredInit};
         T = 10;
         sol = NDSolve[LVmodel, {x[t], y[t]}, {t, 0, T}][[1]]
```

```
In[57]:=
```

- More informative presentation: construct the parametric curve $(x(t), y(t))$

```
In[49]:= {r, a, m, b} = {1, 1, 1, 1}; (* model parameters *)
         PreyDE = x'[t] == r x[t] - a x[t] y[t];
         PredDE = y'[t] == -m y[t] + b x[t] y[t];
         PreyInit = x[0] == 1.5;
         PredInit = y[0] == 0.5;
         LVmodel = {PreyDE, PredDE, PreyInit, PredInit};
         T = 10;
         sol = NDSolve[LVmodel, {x[t], y[t]}, {t, 0, T}][[1]]
```

$\{x(t) \to \text{InterpolatingFunction}[](t), y(t) \to \text{InterpolatingFunction}[](t)\}$

```
In[57]:=
```

- Populations can evolve according to close cycles

```
In[57]:= ParametricPlot[{x[t],y[t]} /. sol,{t,0,T}]
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In[57]:= ParametricPlot[{x[t],y[t]} /. sol,{t,0,T}]
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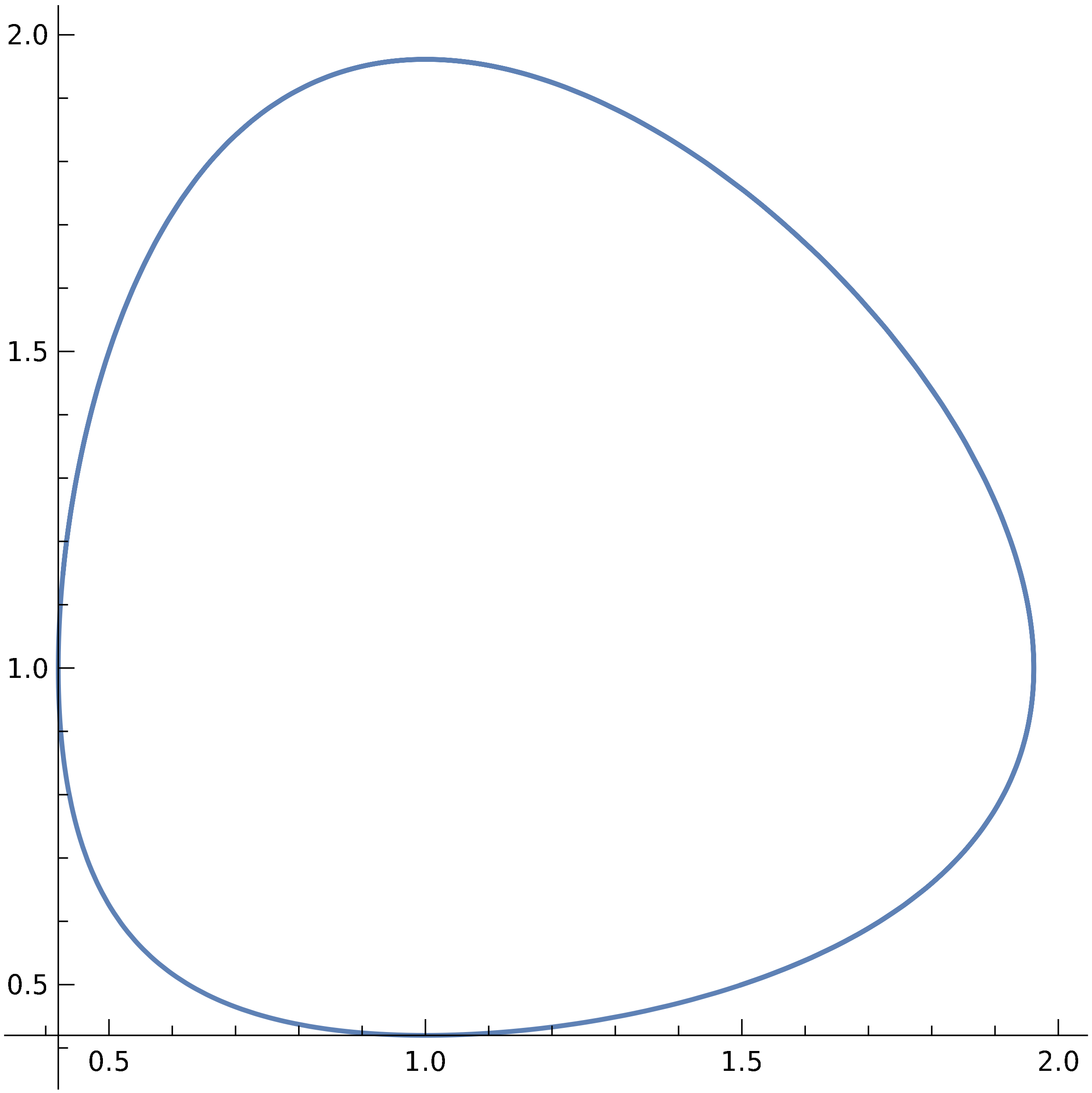

- Consider populations x, y, z representing wolves, sheep, grass

$$x' = x(y-1), y' = y(-1+2z-x), z' = z(2-z-y)$$

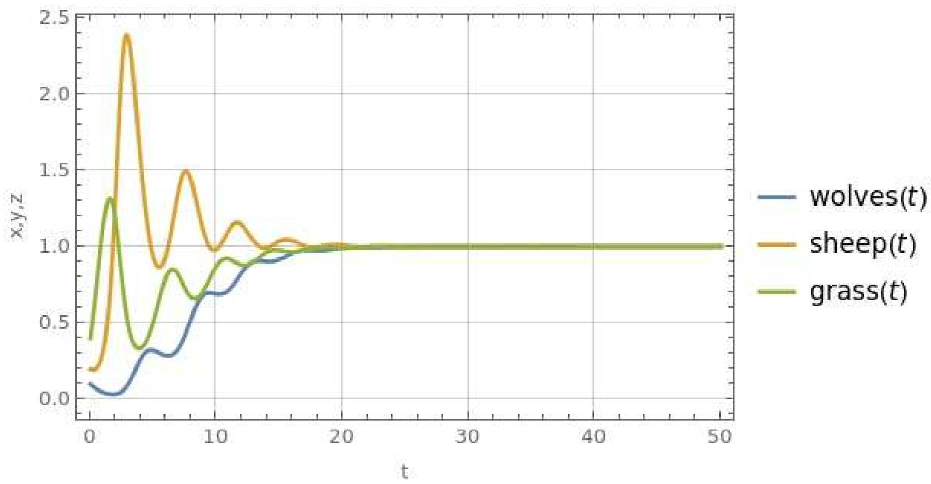

Figure 1. Typical 3 species model solution - equilibrium.