- Age-structure models
  - Age bin models
  - Continuous age models
  - Discrete time models
  - Continuous time models

- Previous models considered the total population $p' = rp$ in Malthus model
- Organisms exhibit different characteristics during a lifetime, e.g., maturation

$$p'(t) = rp(t - \tau)$$

- Changes might be more extensive than the capability to reproduce from a maturation model, e.g., decrease of fertility with age
- A more informative population model would consider age groups
- A common representation is age bins ("Baby boomers", "Generation X"), mathematically, instead of a single population function $p(t)$ consider the population evolution in some choice of N age bins, $p(n,t)$, $n \in \{1, ..., N\}$
- Another representation is to consider age as a continuous parameter $y \in [0, T]$, and describe the population by $p(y,t)$.
- The time t can be considered continuous (leading to ODEs) or discrete (leading to recurrence relations)

- Each population bin satisfies some base model (e.g., Malthus or logistic), but with bin-specific coefficients. Assume $n=0$ corresponds to a bin of new-borns

$$p_0(t+1) - p_0(t) = \sum_{j=1}^{N} f_j p_j(t) - m_0 p_0(t)$$

The above states newborns arise from populations of elders p_j with fecundity f_j, and decrease due to mortality m_0.

- Elder populations satisfy

$$p_n(t+1) - p_n(t) = s_{n-1} p_{n-1}(t) - m_n p_n(t)$$

with s_{n-1} the survival rate of the younger generation and m_n the mortality of age-bin n

- Introduce vector $\boldsymbol{p}(t) = [\ p_0(t)\ \ \dots\ \ p_N(t)\ ]^T \in \mathbb{R}^{N+1}$. Obtain

$$\boldsymbol{p}(t+1) = \begin{bmatrix} 1-m_0 & f_1 & f_2 & \dots & f_N \\ s_0 & 1-m_1 & 0 & \dots & 0 \\ 0 & s_1 & 1-m_2 & \dots & 0 \\ & & & \ddots & \\ & & & & 1-m_N \end{bmatrix} \begin{bmatrix} p_0(t) \\ p_1(t) \\ \vdots \\ p_N(t) \end{bmatrix} = \boldsymbol{L}\boldsymbol{p}(t)$$

- Stable population $\Rightarrow \boldsymbol{p}(t+1) = \boldsymbol{p}(t)$. The stable population distribution is given by eigenvectors of the Leslie matrix $\boldsymbol{L}$

- Choose Malthus base model

$$p_0'(t) = \sum_{j=1}^{N} f_j \, p_j(t) - m_0 \, p_0(t)$$

$$p_n'(t) = s_{n-1} p_{n-1}(t) - m_n \, p_n(t)$$

- Reformulate in matrix terms to obtain a linear ODE system

$$\boldsymbol{p}'(t) = \boldsymbol{L} \boldsymbol{p}(t)$$

- Let the population in vary small age range δy over a small time interval δt be

$$\delta P = p(y,t)\delta y \delta t$$

In this case $p(y,t)$ represents a *population density*.

- For a Malthus type model, the ODE system $\boldsymbol{p}' = \boldsymbol{L}\boldsymbol{p}$ becomes a PDE

$$\frac{\partial p}{\partial y} + \frac{\partial p}{\partial t} = -\mu(y,t)p$$

with μ the overall fertility rate.