- Serially linked neurons
- Wave partial differential equation
- Hodgkin-Huxley space-time model

- The Hodgkin-Huxley ODE system is

$$\begin{aligned}
m' &= \alpha_m(V)(1-m) - \beta_m(V)m \\
n' &= \alpha_n(V)(1-n) - \beta_n(V)n \\
h' &= \alpha_h(V)(1-h) - \beta_h(V)h \\
C_m V' &= \bar{g}_{\mathrm{Na}}\, m^3 h\, (V - E_{\mathrm{Na}}) + \bar{g}_{\mathrm{K}}\, n^4\, (V - E_{\mathrm{K}}) + \\
&\quad g_\ell\, (V - E_\ell) + P(t)
\end{aligned}$$

  where $P(t)$ is the pulse (e.g., synapse) excitation

- When linking neurons:
  - one neuron is excited by a pulse
  - successive neurons are excited by the voltage from previous neuron

- Recall diffusion equation

$$\frac{\partial c}{\partial t} = \alpha \frac{\partial^2 c}{\partial x^2}.$$

The equation is first order in time, second order in space. The different orders of differentiation indicate unresolved phenomena, i.e., microscopic diffusion.

- In contrast, the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

is second order in both space and time. The equal orders of differentiation indicate a fully resolved phenomenon. The solution is of the form

$$u(t,x) = f(x - at) + g(x + at)$$

- Action potentials propagate along an axon at a speed $a = 21$ m/s according to a wave equation if no ion channels are open

$$\frac{\partial^2 V}{\partial t^2} = a^2 \frac{\partial^2 V}{\partial x^2}$$

- Apply Kirchhoff circuit law to a small portion of the axon to obtain

$$\frac{r}{2R}\frac{\partial^2 V}{\partial x^2} - (i_{\mathrm{Na}} + i_{\mathrm{K}} + i_{\ell}) = C_m \frac{\partial V}{\partial t}$$

- Combine with wave equation to obtain for a given position x

$$\frac{r}{2Ra^2}\frac{\partial^2 V}{\partial t^2} - (i_{\mathrm{Na}} + i_{\mathrm{K}} + i_{\ell}) = C_m \frac{\partial V}{\partial t}$$

- Consider J ion channels along an axon at positions x_1, x_2, ..., x_J, with voltages $V_j(t)$, $j = 1, \ldots, J$. Assume equidistant ion channels $x_j = j\Delta$
- Between ion channels the pulse travels according to the wave equation, such that the pulse excitation at position $j+1$ is $P_{j+1} = V_j(t - \Delta/a)$
- Such an approach reduces the partial differential equation describing action potential propagation to a system of ODEs, and is known as a method of lines