



- SIR space-time model
- Graphs
- Calculus on graphs
- Graph Laplacian, diffusion

- Recall coupled PDE SIR model

$$\begin{aligned}S'(t) &= -\beta I(t) S(t) \\I'(t) &= \beta I(t) S(t) - \gamma I(t) \\R' &= \gamma I \\S, I, R: \mathbb{R} &\rightarrow \mathbb{R}\end{aligned}$$

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\beta S (I + \alpha \nabla^2 I) + \delta \nabla^2 S \\ \frac{\partial I}{\partial t} &= \beta S (I + \alpha \nabla^2 I) - \gamma I + \epsilon \nabla^2 I \\ \frac{\partial R}{\partial t} &= \gamma I \\ S, I, R: \mathbb{R} \times \mathbb{R}^d &\rightarrow \mathbb{R}\end{aligned}$$

- Form four-parameter reduced SI model

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\beta S (I + \alpha \nabla^2 I) + \delta \nabla^2 S \\ \frac{\partial I}{\partial t} &= \beta S (I + \alpha \nabla^2 I) - \gamma I \\ S, I: \mathbb{R} \times \mathbb{R}^2 &\rightarrow \mathbb{R}\end{aligned}$$

- Graph: a set of vertices and edges, $G = (V, E)$
- Graph connectivity matrix, $\mathbf{C} \in \mathbb{R}^{m \times m}$, $m = |V|$

$$\mathbf{C} = [C_{ij}]_{1 \leq i, j \leq m}, C_{ij} \neq 0 \text{ if } V_i \text{ connected to } V_j$$

- $C_{ij} \in \{0, 1\}$, simple connectivity data, neighbor of relation, $x_i \sim x_j$
- $C_{ij} \in \mathbb{R}$, additional data, e.g., distance, flux, In general some “weight” associated with edge, hence notation $\mathbf{W} = [w_{ij}]_{1 \leq i, j \leq m}$
- Graphs can represent both:
 - discretizations of continua, e.g., Cartesian grid within $\mathbb{R}^2$
 - disperse data, e.g., positions of cities, counties in the US
- Similar to calculus in $\mathbb{R}, \mathbb{R}^2, \dots$, a calculus can be define for functions defined on a graph

- Differentiation, $f: \mathbb{R} \rightarrow \mathbb{R}$, $g: G \rightarrow \mathbb{R}$

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}, \quad \frac{\partial g(x_i)}{\partial x_j} = \sqrt{w_{ij}} [g(x_j) - g(x_i)]$$

- Gradient, $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, $g: G \rightarrow \mathbb{R}$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right), \quad \nabla_w g(x_i) = \left(\frac{\partial g}{\partial x_j} \right)_{j \sim i}$$

- Divergence, $\mathbf{v}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $F: G \rightarrow \mathbb{R}$, $F_{ij} = F(x_j, x_i)$

$$\nabla \cdot \mathbf{v} = \lim_{|\Omega| \rightarrow 0} \frac{1}{|\Omega|} \oint_{S=\partial\Omega} \mathbf{v} \cdot \mathbf{n} dS, \quad \nabla_w^* F(x_i) = \sum_{x_j \sim x_i} \sqrt{w_{ij}} [F_{ij} - F_{ji}]$$

- Laplacian in $\mathbb{R}^2$, $\nabla^2 f = \nabla \cdot (\nabla f)$
- Laplacian in $G = (V, E)$

$$\Delta_w f(x_i) = \nabla_w^2 f(x_i) = \sum_{x_j \sim x_i} w_{ij} (f(x_j) - f(x_i))$$

- Diffusion in $\mathbb{R}^2$

$$\frac{\partial f}{\partial t} = \alpha \nabla^2 f$$

- Graph diffusion equation

$$\frac{df(x_i)}{dt} = \alpha \nabla_w^2 f(x_i) = \alpha \sum_{x_j \sim x_i} w_{ij} (f(x_j) - f(x_i))$$