$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Guess solution is of form $y(t) \sim e^{rt}$. The guess is motivated by the fact that e^{rt} solve the eigenproblem

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{rt} = r\, e^{rt}$$

Replacing in the equation

$$(r^n + a_{n-1} r^{n-1} + \cdots + a_0) e^{rt} = 0$$

$$r^n + a_{n-1} r^{n-1} + \cdots + a_0 = 0$$

The analogous difference equation is

$$y_{i+k} + a_{k-1} y_{i+k-1} + \ldots + a_0 y_i = 0$$

The solution procedure is analogous, with guess, $y_n \sim r^n$. Replacing, we obtain

$$(r^k + a_{k-1} r^{k-1} + \cdots + a_0) r^i = 0$$

Non-trivial solution is obtained for

$$r^k + a_{k-1} r^{k-1} + \cdots + a_0 = 0$$

1 Analogy between difference and differential equations

Finite difference approximations of derivatives, at equidistant points $t_n = nh$

$$y'(t_n) = y'(nh) \cong \frac{y_{n+1} - y_n}{h} \cong \frac{y_n - y_{n-1}}{h} \cong \frac{y_{n+1} - y_{n-1}}{2h}$$

$$y''(t_n) \cong \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$$

$$y'' + a_1\, y' + a_0\, y = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} + a_1 \frac{y_{n+1} - y_{n-1}}{2h} + a_0\, y_n = 0$$

$$\left(1 + \frac{h a_1}{2}\right) y_{n+2} + (a_0\, h^2 - 2)\, y_{n+1} + \left(1 - \frac{h a_1}{2}\right) y_n = 0$$

Example, choose $h = 1$

$$y'' - 5\, y' + 6\, y = 0$$

the corresponding difference equation is

$$\left(1-\frac{5}{2}\right)y_{n+2}+(6-2)\,y_{n+1}+\left(1-\frac{(-5)}{2}\right)y_n=0$$

$$-\frac{3}{2}\,y_{n+2}+4\,y_{n+1}+\frac{7}{2}\,y_n=0$$

2 Nonlinear difference equations

Logistic map $\{y_n\}$ is a variation of a simple population model $y_{n+1}=a y_n$, in which the fertility to be variable, dependent on current population, $a=\lambda(1-y_n)$ (meant to model enviromental constraints on maximum population)

$$y_{n+1}=\lambda y_n(1-y_n)=\lambda\,(1-y_n)y_n$$