Overview

- Error functions and truncation error
- Round-off error
- Order, rate of convergence estimation

- Standard sequences $\{y_n\}_{n\in\mathbb{N}}$ converge to a scalar, $\lim_{n\to\infty} y_n = y \in \mathbb{R}$
- Often, the object of numerical approximation is a function

$$\{f_n\}_{n\in\mathbb{N}}, \lim_{n\to\infty} f_n = f;\, f, f_n\colon \mathbb{R}\to\mathbb{R}.$$

- Example: Taylor polynomial (*truncation* of Taylor series) of $f\colon \mathbb{R}\to\mathbb{R}$

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n.$$

- The goal of numerical analysis in such cases is to detemine an *error function*

$$f(x) = p_n(x) + r_n(x), r_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x - x_0)^{n+1}$$

- Structure in error function: not converged, *truncation error* dominates.
- Randomness in error function: converged to machine precision, *roundoff error* dominates.

- Typical convergence behavior demonstrated by derivative approximation of

$$g = f', f(x) = e^x - 1, x_0 = 0, g(x_0) = f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

through *finite differences*

$$g_n = \frac{f_n - f(0)}{h_n}, f_n = f(h_n), h_n = 2^{-n}.$$

n	1	2	$\ldots$	N
$h_n = 2^{-n}$	$1/2$	$1/4$	$\ldots$	$1/2^N$
f_n	f_1	f_2	$\ldots$	f_N
$g_n = \frac{f_n - f(0)}{h_n}$	$\frac{f_1 - f(0)}{(1/2)}$	$\frac{f_n - f(0)}{(1/4)}$	$\ldots$	$\frac{f_n - f(0)}{(1/2^N)}$
$d_{n-1} = \lvert g_n - g_{n-1} \rvert$	$-$	$d_1 = \lvert g_2 - g_1 \rvert$	$\ldots$	$d_{N-1} = \lvert g_N - g_{N-1} \rvert$

Table 1. Calculations to construct approximation of derivative for $f(x)$, at $x_0 = 0$.

- From definition

$$\lim_{n\to\infty}\frac{|x_{n+1}-x|}{|x_n-x|^p}=r$$

- Assume that for some n, $d_n=|x_n-x|$

$$d_{n+1}\cong r\,d_n^p\,.$$

- Since d_n is small, take logarithm, $c_n=\log d_n$ and obtain

$$c_{n+1}\cong p\,c_n+\log r\,.$$

- Subtract successive terms $c_n-c_{n-1}\cong p(c_{n-1}-c_{n-2})$, get average slope

$$p\cong\frac{1}{N-3}\sum_{n=3}^{N-1}\frac{c_n-c_{n-1}}{c_{n-1}-c_{n-2}}$$

- Order of convergence definition $\lim_{n\to\infty}\frac{|x_{n+1}-x|}{|x_n-x|^p}=r$ requires knowledge of limit x. Replace with

$$\lim_{n\to\infty}\frac{|x_{n+1}-x_n|}{|x_n-x_{n-1}|^q}=s$$

  (q,s) are estimates of the order, rate of convergence (p,r)
- It is useful to compare order of convergence of approximation sequence $\{x_n\}_{n\in\mathbb{N}}$ to some standard sequences, in particular:
  - power function n^{-p}
  - exponential function r^{-n}
- Sequences that converge like n^{-p} for some p have *algebraic convergence*
- Sequences that converge like r^{-n} for:
  - some $r>1$ have *geometric convergence*
  - any $r>1$ have *supergeometric convergence*