Overview

- Interval partition and approaches to piecewise interpolation
- Spline interpolation:
  - Constant splines
  - Linear splines
  - Quadratic splines
  - Cubic splines
- Spline bases

- Numerical experiments: high-degree polynomial interpolants can diverge
- Idea: for $f\colon [a,b] \to \mathbb{R}$, break up interval $[a,b]$ into smaller pieces

Definition. $\{x_0, x_1, \ldots, x_m\}$ *is a partition of the interval* $[a,b] \subset \mathbb{R}$ *if* $x_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, *satisfy*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_m = b.$$

Definition. *The norm of partition* $X = \{x_0, x_1, \ldots, x_m\}$ *of the interval* $[a,b] \subset \mathbb{R}$ *is*

$$\|X\| = \max_{1 \leqslant i \leqslant m} |x_i - x_{i-1}|.$$

- Define $S_i\colon [x_{i-1}, x_i] \to \mathbb{R}$, $S_i(t)$ polynomial of degree $n \ll m$
- Approaches to piecewise polynomials interpolation:
  - *Splines*: enforce $S_i^{(k)}(x_i) = S_{i+1}^{(k)}(x_i)$ (continuity up to derivative of order k)
  - *Piecewise Lagrange*: further divide $[x_{i-1}, x_i]$ into n intervals, no derivatives

- Simplest case: constant functions $S_i(t) = y_{i-1}$
- Apply polynomial error formula over each subinterval

$$f(t) - S_i(t) = f'(\xi_t)(t - x_{i-1})$$

- Overall

$$|f(t) - p(t)| \leqslant \|f'\|_\infty \|X\|$$

- For equidistant (uniform) partition $x_i = x_0 + i h$, $h = (x_m - x_0)/m$

$$|f(t) - p(t)| \leqslant \|f'\|_\infty h,$$

- The interpolant $p(t)$ converges to $f(t)$ linearly (order of convergence is 1)
- Type of approximation used analytically in construction of integrals from Riemann sums

- A piecewise linear interpolant is obtained by

$$S_i(t) = \frac{t - x_{i-1}}{x_i - x_{i-1}}(y_i - y_{i-1}) + y_{i-1}.$$

- Interpolation error is bounded by

$$|f(t) - p(t)| \leqslant \frac{1}{2}\|f'\|_\infty h^2$$

- Converges as $\mathcal{O}(h^2)$ ("quadratic convergence", more properly algebraic convergence as $\mathcal{O}(1/n^2)$)

- Piecewise quadratic interpolant, $S_i\colon [x_{i-1}, x_i] \to \mathbb{R}$

$$S_i(t) = b_i(t - x_{i-1})^2 + c_i(t - x_{i-1}) + y_{i-1}.$$

- Interpolation condition at left already satisfied $S_i(x_{i-1}) = y_{i-1}$
- Enforce interpolation condition at right

$$S_i(x_i) = b_i h_i^2 + c_i\, h_i = y_i, \quad i = 1, 2, \ldots, n$$

- Only n conditions for $2n$ parameters. Enforce continuity of derivative in interior

$$S_i'(x_i) = 2b_i h_i + c_i = 2b_{i+1} h_{i+1} + c_{i+1} = S_{i+1}'(x_i) \quad i = 1, 2, \ldots, n-1 \ .$$

- Still one condition left to choose. Examples:
  - $S_n'(x_i) = y_n'$ (known end slope)
  - $S_n'(x_i) = S_n'(x_{i-1})$ (constant end-slope)
- To compute b_i, c_i a linear system $\boldsymbol{B}\boldsymbol{s} = \boldsymbol{d}$ is formed and solved for $s_i = y_i'$

$$\boldsymbol{B}\boldsymbol{s}=\boldsymbol{d}$$

$$\boldsymbol{B}=\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{bmatrix}, \boldsymbol{d}=\begin{bmatrix} \frac{2}{h_1}(y_1-y_0)-s_0 \\ \frac{2}{h_2}(y_2-y_1) \\ \vdots \\ \frac{2}{h_n}(y_n-y_{n-1}) \end{bmatrix}, \boldsymbol{s}\in\mathbb{R}^n, \boldsymbol{B}\in\mathbb{R}^{n\times n}.$$

- The interpolation error is bounded by

$$|f(t)-p(t)|\leqslant\frac{1}{2}\|f'\|_\infty h^2,$$

for an equidistant partition, exhibiting algebraic "quadratic" convergence.

- Approach similar to quadratic, but with continuity up to second derivative
- Continuity of curvature (second derivative) important in applications
- The second derivative is linear

$$S_i''(t) = \frac{z_{i-1}}{h_i}(x_i - t) + \frac{z_i}{h_i}(t - x_{i-1}),$$

- Obtain tridiagonal system

$$h_i z_{i-1} + 2(h_i + h_{i-1})z_i + h_{i+1}z_{i+1} = d_i, i = 1, 2, \ldots, n-1.$$

$$d_i = \frac{6(y_{i+1} - y_i)}{h_{i+1}} - \frac{6(y_i - y_{i-1})}{h_i}$$

- End conditions are required to close the system:
  - Zero end-curvature, "natural end conditions": $z_0 = z_n = 0$
  - Curvature extrapolation: $z_0 = z_1$, $z_n = z_{n-1}$
  - Known curvature: $z_0 = f''(x_0)$, $z_n = f''(x_n)$.

- Splines are also linear combinations of basis functions

$$\mathcal{B}_k(t;\boldsymbol{x}) = \{B_{0,k}(t), B_{1,k}(t), \ldots, B_{n,k}(t)\}$$

- The basis functions are non-zero only over a few subintervals

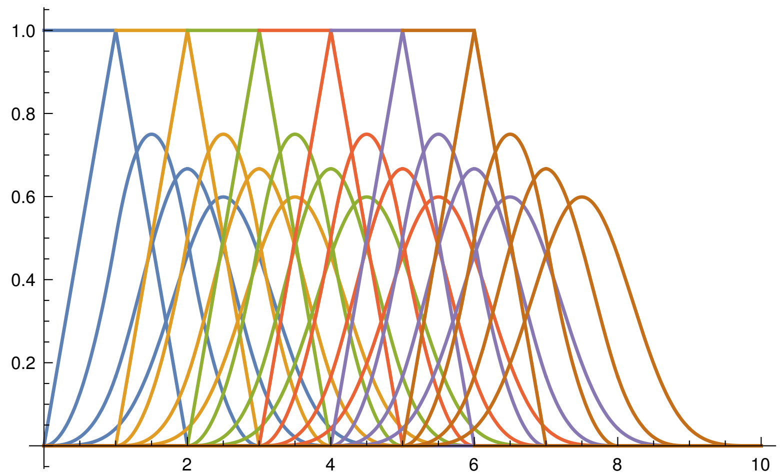

Figure 1. B-spline sets $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$ with $\boldsymbol{x} = [\ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\ ]$