Overview

Motivation: minimize interpolation error

$$f(t) - p_n(t) = \frac{f^{(n+1)}(\xi_t)}{(n+1)!} w(t), w(t) = \prod_{j=0}^{n} (t - x_j)$$

by choosing sample points x_j to ensure $w(t)$ remains small when $t \neq x_j$. We first review the concept of orthonormal bases.

- Orthogonal matrices
- Orthogonal polynomials
- Legendre polynomials
- Chebyshev polynomials

- Linear algebra: most effective bases in $\mathbb{R}^m$ are orthonormal, e.g. column vectors of the identity matrix

$$\boldsymbol{I} = [\ \boldsymbol{e}_1\ \ \boldsymbol{e}_2\ \ \ldots\ \ \boldsymbol{e}_m\ ]$$

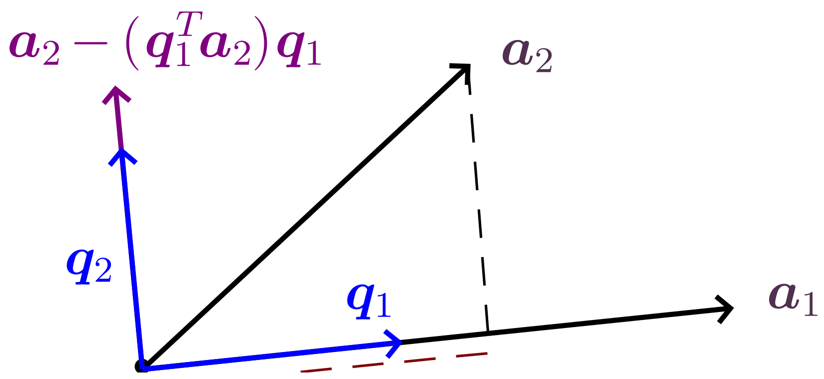
- In general $\boldsymbol{Q} = [\ \boldsymbol{q}_1\ \ \boldsymbol{q}_2\ \ \ldots\ \ \boldsymbol{q}_m\ ] \in \mathbb{R}^{m\times m}$ is an orthogonal matrix if the scalar product of two column vectors satisfies

$$\boldsymbol{q}_i^T \boldsymbol{q}_j = \delta_{ij}, \boldsymbol{Q}^T\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{q}_1^T \\ \boldsymbol{q}_2^T \\ \vdots \\ \boldsymbol{q}_m^T \end{bmatrix} [\ \boldsymbol{q}_1\ \ \boldsymbol{q}_2\ \ \ldots\ \ \boldsymbol{q}_m\ ] = \begin{bmatrix} \boldsymbol{q}_1^T \boldsymbol{q}_1 & \ldots & \boldsymbol{q}_1^T \boldsymbol{q}_m \\ \vdots & \ddots & \vdots \\ \boldsymbol{q}_m^T \boldsymbol{q}_1 & \ldots & \boldsymbol{q}_m^T \boldsymbol{q}_m \end{bmatrix} = \boldsymbol{I}$$

- Recall: vectors are functions on the domain $\{1, 2, \ldots, m\}$, e.g. $e_k(i) = \delta_{ik}$
- The scalar product is expressed as

$$\boldsymbol{q}_i^T \boldsymbol{q}_j = \delta_{ij} = \sum_{l=1}^{m} q_i(l)\, q_j(l)$$

- Assume $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ is of full rank, i.e., has linearly independent columns
- Column vectors of $\boldsymbol{A} = [\ \boldsymbol{a}_1 \ \ \dots \ \ \boldsymbol{a}_m\ ]$ may not be orthonormal, but an orthonormal basis can be obtained by the Gram-Schmidt process (QR - factorization):
  1. Start with an arbitrary direction $\boldsymbol{a}_1$
  2. Divide by its norm to obtain a unit-norm vector $\boldsymbol{q}_1 = \boldsymbol{a}_1 / \|\boldsymbol{a}_1\|$
  3. Choose another direction $\boldsymbol{a}_2$
  4. Subtract off its component along previous direction(s) $\boldsymbol{a}_2 - (\boldsymbol{q}_1^T \boldsymbol{a}_2)\boldsymbol{q}_1$
  5. Divide by norm $\boldsymbol{q}_2 = (\boldsymbol{a}_2 - (\boldsymbol{q}_1^T \boldsymbol{a}_2)\boldsymbol{q}_1) / \|\boldsymbol{a}_2 - (\boldsymbol{q}_1^T \boldsymbol{a}_2)\boldsymbol{q}_1\|$
  6. Repeat the above



- Extend scalar product $\boldsymbol{q}_i^T \boldsymbol{q}_j = \delta_{ij} = \sum_{l=1}^{m} q_i(l)\, q_j(l)$ to $f, g \colon [a, b] \to \mathbb{R}$

$$(f, g) = \int_a^b \omega(t)\, f(t)\, g(t)\, \mathrm{d}t$$

- The above is a (real-valued) a function *inner product* if:
  1. $(f, g) = (g, f)$
  2. $(a f + b g, h) = a(f, h) + b(g, h)$
  3. $(f, f) > 0$ and $(f, f) = 0 \Rightarrow f = 0$
- Gram-Schmidt process can be applied to sets of functions $\{f_1, \ldots, f_m\}$ to obtain an orthonormal set $\{g_1, \ldots, g_m\}$ with respect to a specific inner product using norm $\|f\| = (f, f)^{1/2}$
  1. $g_1 = f_1 / \|f_1\|$
  2. $h = f_2 - (f_2, g_1)\, g_1$
  3. $g_2 = h / \|h\|$
  4. $h = f_3 - (f_3, g_1) g_1 - (f_3, g_2) g_2$
  5. $g_3 = h / \|h\|$, and continue

- Orthogonal polynomials play an important role in numerical methods, they furnish a more effective basis than the monomial basis $\{1, t, t^2, \ldots\}$ used in the Taylor series
- For scalar product with weight $\omega(t) = 1$

$$(f, g) = \int_{-1}^{1} f(t)\, g(t)\, \mathrm{d}t$$

applying the Gram-Schmidt process leads to the Legendre polynomials

Figure 1. Comparison of monomial basis (left) to Legendre polynomial basis (right).

- Gram-Schmidt process applied to $\{1, t, t^2, \ldots\}$ with scalar product

$$(f, g) = \int_{-1}^{1} \frac{f(t)\, g(t)}{\sqrt{1-t^2}}\, \mathrm{d}t$$

leads to the Chebyshev polynomials, $T_n(t)$, $\omega(t) = (1-t^2)^{-1/2}$

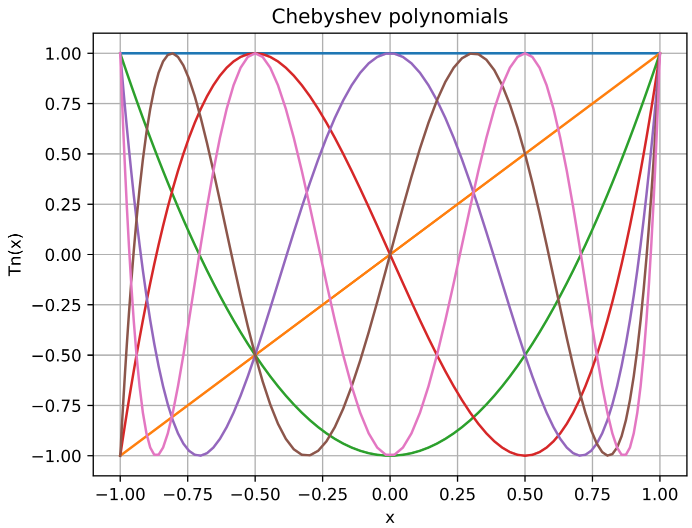


Figure 2. Chebyshev polynomials. Easily distinguishable. with values in $[-1, 1]$, and roots clustered toward the interval endpoints