Overview

Motivation: minimize interpolation error

$$f(t) - p_n(t) = \frac{f^{(n+1)}(\xi_t)}{(n+1)!} w(t), w(t) = \prod_{j=0}^{n} (t - x_j)$$

Here, the best possible behavior of $w(t)$ is identified.

- Trigonometric expression of Chebyshev weight scalar product
- Chebyshev recurrence relation
- Roots and extrema of Chebyshev polynomials
- Minimal inf-norm property of Chebyshev polynomials
- Runge function on Chebyshev grid

- Chebyshev polynomials: obtained by orthonormalization of $\{1, t, t^2, \ldots\}$ w.r.t

$$(f, g) = \int_{-1}^{1} \frac{f(t)g(t)}{\sqrt{1-t^2}} \, \mathrm{d}t.$$

- The Gram-Schmidt process can be applied, but a quicker route is:

$$t = \cos\theta \Rightarrow \mathrm{d}t = -\sin\theta \, \mathrm{d}\theta = -\left(\sqrt{1-\cos^2\theta}\right)\mathrm{d}\theta = -\left(\sqrt{1-t^2}\right)\mathrm{d}\theta \Rightarrow$$

$$(f, g) = \int_0^{\pi} f(t(\theta))\, g(t(\theta))\, \mathrm{d}\theta = \int_0^{\pi} F(\theta) G(\theta)\, \mathrm{d}\theta.$$

- Consider $F(\theta) = \cos(j\theta)$, $G(\theta) = \cos(k\theta)$ and compute

$$\int_0^{\pi} \cos(j\theta) \cos(k\theta)\, \mathrm{d}\theta = c_j\, \delta_{jk}, \, c_0 = \pi\,, \, c_j = \frac{\pi}{2} \text{ for } j > 0$$

- Introduce notation $T_n(t(\theta)) = \cos(n\theta)$ (*un-normalized Chebyshev polynomial*)
- Observe: $T_0(t) = 1$, $T_1(t) = \cos\theta = t$,

$$T_{n+1}(t(\theta)) + T_{n-1}(t(\theta)) = \cos[(n+1)\theta] + \cos[(n+1)\theta] = 2\cos\theta\cos(n\theta)$$

- The above trigonometric identity leads to the recurrence relationship

$$T_{n+1}(t) = 2tT_n(t) + T_{n-1}$$

- Since $T_0(t) = 1$, $T_1(t) = t$, recurrence relationship implies $T_n(t)$ are polynomials

$$T_2(t) = 2t^2 - 1, T_3(t) = 4t^3 - 3t, T_4(t) = 8t^4 - 8t^2 + 1, \ldots$$

- Unit-norm Chebyshev polynomials are $T_n(t)/c_n$, $c_0 = \pi$, $c_j = \frac{\pi}{2}$, $j > 1$

$$(T_m, T_n) = \int_{-1}^{1} \frac{T_m(t)T_n(t)}{\sqrt{1-t^2}}\,\mathrm{d}t = \int_0^{\pi} \cos(m\theta)\cos(n\theta)\,\mathrm{d}\theta = c_n\,\delta_{mn}.$$

Roots and extrema Chebyshev polynomials

- Roots of Chebyshev polynomials:

$$T_n(t(\theta)) = \cos(n\theta) = 0 \Rightarrow \theta_j = \frac{\pi(2j+1)}{2n}, z_j = \cos\theta_j, j = 0, 1, \ldots, n-1$$

- Local extrema of Chebyshev polynomials:

$$T_n'(t) = -\frac{1}{\sin\varphi} \cdot \frac{\mathrm{d}}{\mathrm{d}\varphi}[T_n(t(\varphi))] = -\frac{1}{\sin\varphi} \cdot \frac{\mathrm{d}\cos(n\varphi)}{\mathrm{d}\varphi} = n\frac{\sin(n\varphi)}{\sin\varphi} = 0 \Rightarrow$$

$$\sin(n\varphi) = 0 \Rightarrow \varphi_j = \frac{j\pi}{n}, x_j = \cos\varphi_j = \cos\frac{j\pi}{n}, j = 1, 2, \ldots, n-1$$

Include end point extrema: $x_j = -\cos(j\pi/n)$, $j = 0, 1, \ldots, n$

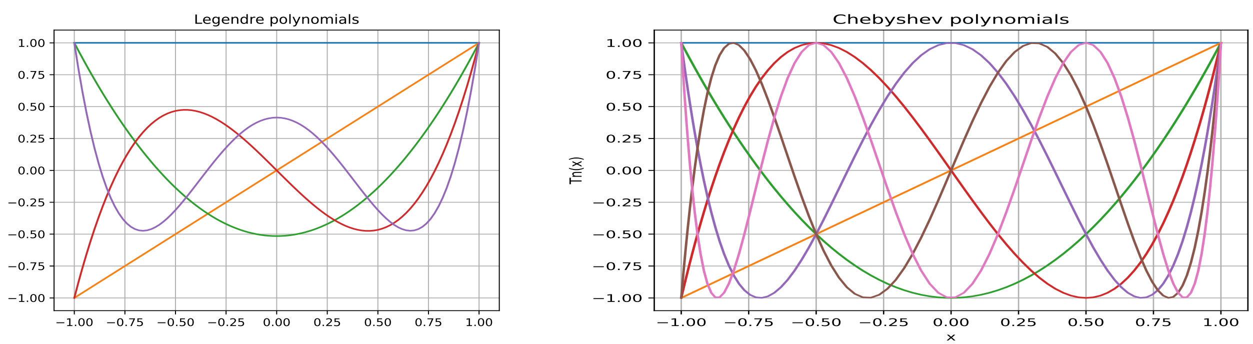

Figure 1. Legendre and Chebyshev polynomials.

- Chebyshev polynomials and the behavior of $w(t) = \prod_{j=0}^{n} (t - x_j)$
- Introduce inf-norm $\| f \|_\infty = \max_{-1 \leqslant t \leqslant 1} |f(t)|$
- Monic form of Chebyshev polynomials: $P_0(t) = 1$, $P_n(t) = 2^{1-n} T_n(t)$

$$T_0(t) = 1, T_1(t) = t, T_2(t) = 2t^2 - 1, T_3(t) = 4t^3 - 3t, T_4(t) = 8t^4 - 8t^2 + 1, \ldots$$

  Coefficient of term of n^{th} degree in $P_n(t)$ is one (*monic polynomial*) like $w(t)$

$$\|P_0(t)\| = 1, \|P_n(t)\|_\infty = 2^{1-n}, n > 0$$

- Intuitively: Runge function overshoot of interpolant near interval endpoints

$$f(t) = \frac{1}{1 + (5t)^2}, f(t) - p_n(t) = \frac{f^{(n+1)}(\xi_t)}{(n+1)!} w(t), w(t) = \prod_{j=0}^{n} (t - x_j)$$

  is not due to $f^{(n+1)}(\xi_t)$ (very smooth), but to $w(t)$, i.e., choice of x_j
- What monic polynomial has the smallest inf-norm? The Chebyshev polynomial.

Theorem. $p\colon[-1,1]\to\mathbb{R}$, *monic polynomial of degree* n *has a inf-norm lower bound*

$$\|p\|_\infty = \max_{-1\leqslant t\leqslant 1} |p(t)| \geqslant 2^{1-n}.$$

Proof. *By contradiction, assume the monic polynomial* $p\colon[-1,1]\to\mathbb{R}$ *has* $\|p\|_\infty < 2^{1-n}$. *Construct a comparison with the Chebyshev polynomials by evaluating* p *at the extrema* $x_j = \cos(j\pi/n)$,

$$(-1)^j p(x_j) \leqslant |p(x_j)| < 2^{1-n} = (-1)^j P_n(x_j) = (-1)^j 2^{1-n} T_n(x_j).$$

Since the above states $(-1)^j p(x_j) < (-1)^j P_n(x_j)$ *deduce*

$$(-1)^j [p(x_j) - P_n(x_j)] < 0, \text{for } j = 0, 1, \ldots, n \tag{1}$$

However, p, P_n *both monic implies that* $p(x_j) - P_n(x_j)$ *is a polynomial of degree* $n-1$ *that would change signs* n *times to satisfy (1), and thus have* n *roots contradicting the fundamental theorem of algebra.* □

- For polynomial interpolant of degree n of $f\colon[-1,1]\to\mathbb{R}$ choose error term

$$w(t)=\prod_{j=0}^{n}(t-z_j)=T_{n+1}(t), z_j=\cos\theta_j, \theta_j=\frac{\pi(2j+1)}{2(n+1)}, j=0,1,\ldots,n.$$

$$\mathcal{Z}=\{(z_j,f(z_j)), j=0,1,\ldots,n\}$$

- Alternatively, roots of $T_n'(t)=nU_{n-1}(t)$, $U_{n-1}(t)=\sin(n\theta)/\sin\theta$ and ±1

$$w(t)=\prod_{j=0}^{n}(t-x_j), x_j=-\cos\varphi_j, \varphi_j=\frac{j\pi}{n}, j=0,1,\ldots,n$$

$$\mathcal{D}=\{(x_j,f(x_j)), j=0,1,\ldots,n\}$$

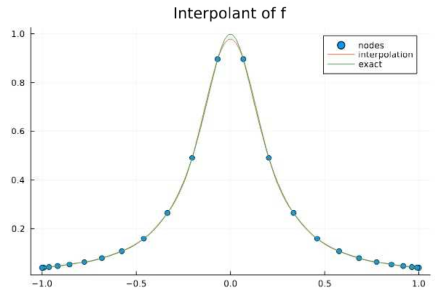

Figure 2. Chebyshev grid interpolant of degree 23

- *Numerical methods* suggested an orthogonal basis set $\{T_0, T_1, \dots\}$
- *Numerical analysis* has furnished an optimal set of sample points x_j or z_j
- Still to study: *computational complexity*
- Naïve approach: $f(t) \cong p(t) = \sum_{j=0}^{n} c_j T_j(t)$, apply interpolation conditions

$$y_j = f(x_j) = p(x_j) \Rightarrow [\; T_0(\boldsymbol{x}) \;\; T_1(\boldsymbol{x}) \;\; \dots \;\; T_n(\boldsymbol{x}) \;] \boldsymbol{c} = \boldsymbol{y} \Leftrightarrow \boldsymbol{A}\boldsymbol{c} = \boldsymbol{y}$$

  obtaining a linear system with a full matrix at computational cost $\mathcal{O}(n^3/3)$
- Recall: polynomial interpolant is *unique*. Use Lagrange barycentric form

$$p(t) = \sum_{j=0}^{n} y_j \frac{w_j}{t - x_j} \Bigg/ \sum_{j=0}^{n} y_j \frac{w_j}{t - x_j}$$

- When sample points are $x_j = -\cos\varphi_j$, $\varphi_j = \frac{j\pi}{n}$, $j = 0, 1, \dots, n$, weights are

$$w_j = (-1)^j, j = 1, \dots, m - 1, w_0 = 1/2, w_n = (-1)^n/2$$

  favoring data set $\mathcal{D} = \{(x_j, f(x_j))\}$ over $\mathcal{Z} = \{(z_j, f(z_j))\}$, $j = 0, 1, \dots, n$.