MATH566 Midterm Exercises

1. Compute the order and rate of convergence for the following sequences $\{x_n\}_{n\in\mathbb{N}}$:

   a) $x_n = (n-1)/(n^2+2)$

   b) $x_n = \sqrt{n+1} - \sqrt{n}$

   c) $x_n = (\sin n)/n$

   d) $x_n = (3n^2-1)/(7n^2+n+2)$

2. Consider $f(x) = x^2 - 4x + a$.

   a) What are the zeros of f (i.e., solutions of $f(x)=0$) for $a=4$?

   b) What are the zeros of f for $a = 4 - 10^{-8}$?

   c) What are the zeros of f for $a = 4 + 10^{-8}$?

   d) Consider the mathematical problem of finding the zeros of f for given a? Based upon the above results what is the conditioning of this problem?

3. Identify values of x where f might exhibit loss of accuracy when evaluated using floating point numbers. Find a different formulation of f (e.g., using algebraic or trigonometric identities) that would alleviate the loss of accuracy.

$$\begin{array}{ll} f(x) = 1+\cos x & f(x) = e^{-x} + \sin x - 1 \\ f(x) = \ln x - \ln(1/x) & f(x) = \sqrt{x^2+1} - \sqrt{x^2+4} \\ f(x) = 1 - 2\sin^2 x & f(x) = \ln\left(x + \sqrt{x^2+1}\right) \\ f(x) = x - \sin x & f(x) = \ln x - 1 \end{array}$$

4. Apply two iterations of Newton's method to approximate the zeros of f, $f(x)=0$, for

$$\begin{array}{ll} f(x) = \ln(1+x) - \cos x & f(x) = x^5 + 2x - 1 \\ f(x) = e^{-x} - x & f(x) = \cos x - x \end{array}$$

5. Apply two iterations of the secant method to approximate the zeros of above listed functions f.

6. Consider $f(x) = x(x^2-1)$. Sketch this function. Graphically show the path of successive Newton iterates. Does Newton's method converge from any starting point?

7. Consider $f(x) = x^2 - 1$. Sketch this function. Graphically show the path of successive Newton iterates. Does Newton's method converge from any starting point?

8. Apply Newton's method to find zeros of the systems

$$\begin{cases} e^x - y = 0 \\ e y^2 - 6x - 4 = 0 \end{cases} \qquad \begin{cases} x^2 + y^2 = 5 \\ x^3 + y^3 = 2 \end{cases}$$

$$\begin{cases} x^3 + 10x - y - 5 = 0 \\ x + y^3 - 10y + 1 = 0 \end{cases} \qquad \begin{cases} 2x - \cos y = 0 \\ 2y - \sin x = 0 \end{cases}$$

9. Formulate gradient descent for the above systems and carry out one iteration from a suitable initial approximation of the zeros.

10. Consider the data $\mathcal{D} = \{(0,2), (1,-1), (2,4)\}$.

    a) Find the interpolating polynomial in the monomial basis.

    b) Construct the Lagrange form of the interpolating polynomial, and expand it to recover the monomial form.

    c) Construct the table of divided differences, the Newton form of the interpolating polynomial, and expand it to recover the monomial form.

11. Construct an optimal quadratic polynomial to approximate $f: [-1,1] \to \mathbb{R}$, $f(x) = e^x$.

12. Construct an optimal quartic polynomial to approximate $f: [1,4] \to \mathbb{R}$, $f(x) = 1/x$.