Overview

- Method of moments approach to quadrature formulas
- Adaptive quadrature: recursive quadrature
  - Idea: repeatedly sample intervals of high quadrature error
  - Implementation: recursion (L20.jl)

- $f: \mathbb{R} \to \mathbb{R}$ known through data set (f sample) $\mathcal{D} = \{(x_i, f_i), i = 0, 1, \ldots, n\}$
- Recall polynomial interpolant technique $f(t) \cong p(t) = \sum_{i=0}^{n} f_i \ell_i(t)$

$$\int_a^b f(t)\,\mathrm{d}t \cong \sum_{i=0}^{n} \left(\int_a^b \ell_i(t)\,\mathrm{d}t \right) f_i = \sum_{i=0}^{n} w_i f_i$$

- Alternative approach: impose exact quadrature for members of a basis set
- Set a simple, predefined integration domain, e.g., $[0,1]$, $x_i = i h$, $h = 1/n$
- Monomial basis set: $\mathcal{M} = \{1, t, t^2, \ldots\}$ conditions

$$\begin{aligned}
f(t) = 1: &\quad \int_0^1 1\,\mathrm{d}t &= 1 &= \sum_{i=0}^{n} w_i \\
f(t) = t: &\quad \int_0^1 t\,\mathrm{d}t &= \frac{1}{2} &= h \sum_{i=0}^{n} w_i\, i \\
f(t) = t^2: &\quad \int_0^1 t^2\,\mathrm{d}t &= \frac{1}{3} &= h \sum_{i=0}^{n} w_i\, i^2 \\
&\ldots.
\end{aligned}$$

- Solve above system to find weights w_i

- $f: [a, b] \to \mathbb{R}$ might exhibit regions of slow/rapid variation
- Equidistant sampling over integration domain $[a, b]$ is inefficient
- *Adaptive quadrature*: place sampling points preferentially to minimize error
- Recursive quadrature:
  - Consider $I_{\mathrm{ab}} = \int_a^b f(t)\,\mathrm{d}t$, and let $Q_{\mathrm{ab}}(f)$ be some quadrature rule
  - Compare $Q_{\mathrm{ab}}(f)$ to $Q_{\mathrm{ac}}(f) + Q_{\mathrm{cb}}(f)$ with $a \leqslant c \leqslant b$, e.g., $c = (a+b)/2$

$$e = \frac{|Q_{\mathrm{ac}} + Q_{\mathrm{cb}} - Q_{\mathrm{ab}}|}{|Q_{\mathrm{ab}}|}$$

  - If error is acceptable, return value, otherwise further subdivide $[a, c]$, $[c, b]$
- See L20.jl