Overview

- Sequence convergence acceleration
- Aitken acceleration
- Composite trapezoid + Aitken acceleration = Romberg quadrature

- Recall definitions that characterize sequence convergence

Definition. $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x *with rate* $r\in(0,1)$ *and order* p *if*

$$\lim_{n\to\infty}\frac{|x_{n+1}-x|}{|x_n-x|^p}=r. \tag{1}$$

- If p is known, it can be used to construct a faster converging sequence $\{y_n\}$
- Write $x_n-x\cong r(x_{n-1}-x)^p$, $x_{n-1}-x\cong r(x_{n-2}-x)^p$, and take ratio

$$\frac{x_n-\bar{x}}{x_{n-1}-\bar{x}}=\left(\frac{x_{n-1}-\bar{x}}{x_{n-2}-\bar{x}}\right)^p. \tag{2}$$

- If p, x_{n-2}, x_{n-1}, x_n are known, then the above can be solved for $\bar{x}$
- The solution $\bar{x}$ is taken as the n^{th} term $y_n=\bar{x}$ in a faster converging sequence
- Since it is based upon assumed behavior, this is called an *extrapolation*, understood as "going outside realm of known data"
- Other extrapolation example: Let $p_m(t)$ interpolate data $\mathcal{D}=\{(ih, y_i), i=0,1,\ldots,m\}$. $p_m(jh)$ for $j\in[0,m]$ is interpolation, for $j>m$ is an extrapolation.

- Extrapolation for $p = 1$ is known as Aitken acceleration

$$\frac{x_n - \bar{x}}{x_{n-1} - \bar{x}} = \frac{x_{n-1} - \bar{x}}{x_{n-2} - \bar{x}}$$

- Obtain

$$x_n x_{n-2} - (x_n + x_{n-2})\bar{x} = x_{n-1}^2 - 2x_{n-1}\bar{x} \Rightarrow \bar{x} = \frac{x_n x_{n-2} - x_{n-1}^2}{x_n - 2x_{n-1} + x_{n-2}}.$$

- Set $y_n = \bar{x}$, rewritten as a correction to the x_n term

$$y_n = x_n - \frac{(x_n - x_{n-1})^2}{x_n - 2x_{n-1} + x_{n-2}}$$

- Example: L21.jl shows acceleration of forward finite difference approximation

$$f_0' \cong \frac{f(x_0 + h_n) - f(x_0)}{h_n}, h_n = 2^{-n}$$

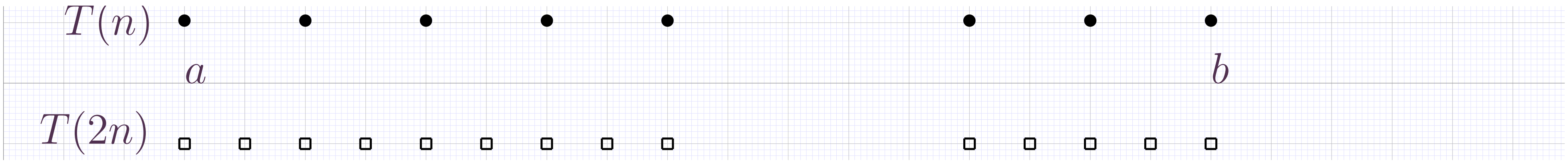
- Consider problem of computing $I_{\mathrm{ab}}(f) = \int_a^b f(t)\,\mathrm{d}t$ by composite trapezoid with uniform partition of interval $[a,b]$, $x_i = x_0 + i\,h_n$, $h_n = (b-a)/n$

$$I_{\mathrm{ab}}(f) \cong Q_{\mathrm{ab}}(f) = T(n) = (h_n/2)\,[f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n]$$

- Introduce notation $\sum''$ to indicate that first, last terms in sum are halved

$$\sum_{i=0}^{n}{}'' f_i = \frac{1}{2} f_0 + f_1 + f_2 + \cdots + f_{n-1} + \frac{1}{2} f_n,\; T(n) = h_n \sum_{i=0}^{n}{}'' f_i$$

- Recall: computational complexity of numerical quadrature given by number f evaluations. Consider $T(n)$,$T(2n)$, and let $f_i^{(n)} = f(i\,h_n)$



$$\begin{array}{lcllllll} T(n) & = & h_n \cdot & (& f_0^{(n)} + & & f_1^{(n)} + \cdots & + \; f_n^{(n)} \;) \\ T(2n) & = & h_{2n} \cdot & (& f_0^{(n+1)} + & f_1^{(n+1)} + & f_2^{(n+1)} + \ldots & + \; f_{2n}^{(2n)} \;) \end{array}$$

- Many common points: exploit this to reduce f evaluations

- Consider $T(n)$ to have been computed, e.g.,

$$T(1) = \frac{b-a}{2}(f(a) + f(b))$$

- Let $h = (b-a)/(2n)$, express $T(2n)$ in terms of previously computed $T(n)$

$$T(2n) = \frac{1}{2}T(n) + h\sum_{i=1}^{n} f_{2i-1}, f_j = f(jh), x_j = a + jh, j = 0, \ldots, 2n.$$

- The above allows construction of sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n = T(n)$.
- Recall composite trapezoid error formula

$$e_n = |I_{\mathrm{ab}}(f) - T(n)| \leqslant \|f''\|_\infty \frac{b-a}{12} h_n^2$$

of second-order accuracy

- Idea #1: apply extrapolation acceleration to $\{x_n\}$ sequence

$$T(n) - I_{\mathrm{ab}} \cong \|f''\|_\infty \frac{b-a}{12} h_n^2, T(2n) - I_{\mathrm{ab}} \cong \|f''\|_\infty \frac{b-a}{12} h_{2n}^2 \Rightarrow$$

$$\frac{T_n - I_{\mathrm{ab}}}{T_{2n} - I_{\mathrm{ab}}} = 4 \Rightarrow I_{\mathrm{ab}} = T(2n) + \frac{1}{4-1}[T(2n) - T(n)]$$

- Take $y_n = I_{\mathrm{ab}} = T(2n) + \frac{1}{4-1}[T(2n) - T(n)]$, which should now be faster-convergent, (third-order) sequence
- Idea #2: repeatedly apply extrapolation acceleration to $\{y_n\}$ sequence
- Introduce $R(n,q)$, n = number of subintervals, q = number of times extrapolation acceleration has been applied
- Romberg relation

$$R(n,q) = R(n,q-1) + \frac{1}{4^q-1}[R(n,q-1) - R(n-1,q-1)]$$

- Organize computations as a table

$$
\begin{array}{llllll}
\text{Resolution} & \text{Extrapolation steps} \rightarrow & & & & \\
\downarrow & R(0,0) & & & & \\
 & R(1,0) & R(1,1) & & & \\
 & R(2,0) & R(2,1) & R(2,2) & & \\
 & \vdots & \vdots & \vdots & \ddots & \\
 & R(n,0) & R(n,1) & R(n,2) & \ldots & R(n,q)
\end{array}
$$

- Choose n high enough to ensure *resolution*, i.e., function variation is captured
- Choose moderate q, $q \leqslant 4$, to avoid loss of precision due to floating point subtraction