Overview

- Runge-Kutta methods
- Analysis of linear multistep methods
- Boundary locus method

- An ODE is an equality between two operators $D = \frac{\mathrm{d}}{\mathrm{d}t}$, $F = f(t, .)$ acting on y

$$\frac{\mathrm{d}}{\mathrm{d}t} y = f(t, y)$$

- Numerical methods have been introduced to approximate:
  - $D = \frac{\mathrm{d}}{\mathrm{d}t}$ (Euler, leapfrog)
  - f, after integration over a time step $[t_i, t_{i+1}]$, $h = t_{i+1} - t_i$, Adams methods
- Yet another alternative technique to obtain a numerical method is to seek a weighted average of the slopes over a time step

$$y_{i+1} = y_i + \sum_{j=1}^{s} w_j k_j$$

$$k_j = h f\left(t_i + \alpha_j h, y_i + \sum_{l=1}^{j-1} \beta_{jl} k_l \right)$$

- The parameters w_j (weights), and α_j, β_{jl} (evaluation points) are found by Taylor series expansions

- A very widely used method is RK4

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h\,f(t_i, y_i)$$

$$k_2 = h\,f(t_i + h/2, y_i + k_1/2)$$

$$k_3 = h\,f(t_i + h/2, y_i + k_2/2)$$

$$k_4 = h\,f(t_i + h, y_i + k_3)$$

- As suggested by the name, RK4 is fourth order of accuracy over a finite interval $[0, T]$ $\varepsilon = \mathcal{O}(h^4)$, and has a one-step error of fifth order $\tau = \mathcal{O}(h^5)$

- A-B, A-M schemes are examples of *linear multistep methods* (LMMs)
- Applied to $y' = \lambda y$ an LMM leads to

$$\sum_{k=0}^{s} a_k y_{i+k} = z \sum_{k=0}^{s} b_k y_{i+k}, z = \lambda h$$

- The above is a *finite difference equation*. Guess solutions of form $y_n = r^n$, and obtain a *characteristic equation* of form

$$\pi(r; z) = \rho(r) - z\sigma(r) = 0$$

with $\rho(r), \sigma(r)$ the polynomials

$$\rho(r) = \sum_{k=0}^{s} a_k r^k, \sigma(r) = \sum_{k=0}^{s} b_k r^k.$$

- Consistency requires $\rho(1) = 0$, $\rho'(1) - \sigma(1) = 0$.
- Stability requires roots r_i of $\pi(r; z)$ to satisfy $|r_i| < 1$.

Boundary locus method

- An LMM method is stable if the roots of $\pi(r;z) = \rho(r) - z\sigma(r)$ are less than 1
- Consider the limitng case in which a root is of magnitude 1
- Roots can be complex, write $\zeta = e^{i\theta}$, $\pi(e^{i\theta};z) = \rho(e^{i\theta}) - z\sigma(e^{i\theta}) = 0 \Rightarrow$

$$z(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}$$

- As $\theta \in [0, 2\pi]$ a locus of marginally stable $z = \lambda h$ values is graphed in $\mathbb{C}$

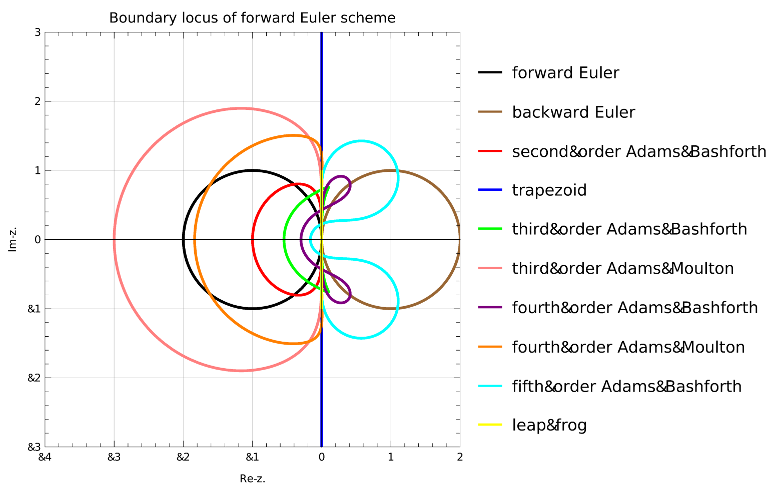

Figure 1. Boundary loci of various methods