MATH590: Approximation in $\mathbb{R}^d$

Abstract

The methods of mathematical analysis are used to process data acquired from cell-phone accelerometers and gyroscopes to: (1) reconstruct the trajectory of the person carrying the cell-phone along a pre-determined path, and (2) determine data features that might identify the person.

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1 Kinematics of rigid body motion

Some basic theory on the kinematics of rigid body motion is necessary to process the acceleration data obtained from the Physics Toolbox Android application. It is assumed that the cell phone and walker form a rigid body, i.e., the cell phone is held rigidly at constant orientation. Note that this hypothesis is likely to be affected by natural flexural motions of the wrist and elbow.

1.1 Reference frames

Describing the kinematics (i.e., motion without consideration of forces) requires consideration of two reference frames:

Absolute reference frame. A fixed coordinate system, i.e., with origin at the starting point of the walk and fixed axis orientations.

Relative reference frame. A moving coordinate system, carried by and centered on the sensor (i.e., cell-phone) motion, with variable axis orientations.

The notation for kinematic quantitites in the two reference systems is presented in Table 1, with a depiction represented in Figure 1. In particular, $\boldsymbol{\Theta}, \boldsymbol{\Omega}$ are the orientation, angular velocity of the relative system expressed as rotation angles, angular velocities around the axes of the absolute system $\mathbf{E}$. The orientation expressed as rotation angles in the relative coordinate system $\mathbf{U}$ are denoted by $\boldsymbol{\theta}$, and $\boldsymbol{\omega}$ are the corresponding angular velocities.

Reference frame	Absolute	Relative	Relationships
Basis vectors	$\mathbf{E} = (\ \boldsymbol{e}_1\ \ \boldsymbol{e}_2\ \ \boldsymbol{e}_3\ )$	$\mathbf{U} = (\ \boldsymbol{u}_1\ \ \boldsymbol{u}_2\ \ \boldsymbol{u}_3\ )$	$\mathbf{U} = \mathbf{R}_{(\varphi,I)}\,\mathbf{E}$
Origin	$\boldsymbol{O} = (0,0,0)$	$\boldsymbol{X}(t)$	
Orientation	$\boldsymbol{\Theta}$	$\boldsymbol{\theta}$	
Linear velocity		$\boldsymbol{V} = \dot{\boldsymbol{X}}$	
Angular velocity	$\boldsymbol{\Omega} = \dot{\boldsymbol{\Theta}}$	$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}}$	
Linear acceleration		$\dot{\boldsymbol{V}} = \ddot{\boldsymbol{X}}$	
Angular acceleration	$\dot{\boldsymbol{\Omega}} = \ddot{\boldsymbol{\Theta}}$	$\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}}$	

Table 1. Kinematic quantitites in the absolute, relative reference frames

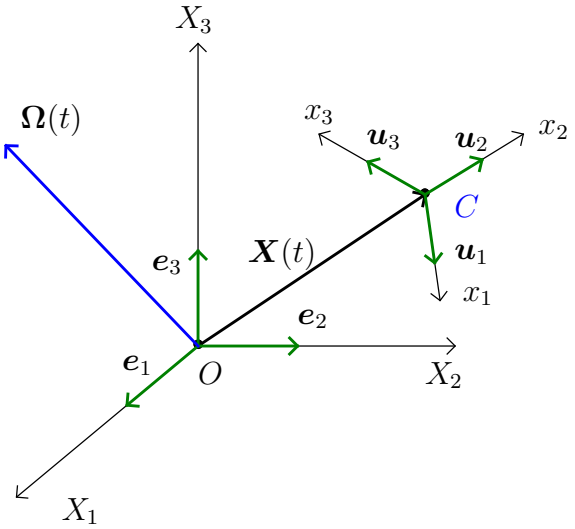

Figure 1. Absolute reference system $(O, \mathbf{E})$, cell phone position C, relative reference system $(C, \mathbf{U})$

1.2 Matrix representation of rotations

During motion, the orientation of the relative reference frame changes due to rotation of the walker. There exists some axis $\boldsymbol{l}$ and angle of rotation φ around the axis $\boldsymbol{l}$ that transforms the original orientation $\mathbf{E}$ into the current orientation of the relative coordinate system $\mathbf{U}$,

$$\mathbf{U} = \mathbf{R}_{(\varphi,\boldsymbol{l})}\,\mathbf{E}. \tag{1}$$

The Euler representation of a rotation matrix is

$$\mathbf{R}_{(\varphi,\boldsymbol{l})} = \cos\varphi\,(\mathbf{E} - \boldsymbol{l}\otimes\boldsymbol{l}) - \sin\varphi\,(\boldsymbol{\epsilon}\boldsymbol{l}) + \boldsymbol{l}\otimes\boldsymbol{l}, \tag{2}$$

where $\boldsymbol{\epsilon}$ is the completely anti-symmetric Levi-Civita tensor with components

$$\epsilon_{ijk} = \text{sign}\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}, \tag{3}$$

the sign of the permutation of $(1,2,3)$ into (i,j,k). The contraction of the rank-3 Levi-Civita tensor with a vector results in a rank-2 tensor (i.e., a matrix)

$$\boldsymbol{\epsilon}\boldsymbol{l} = \begin{pmatrix} 0 & l_3 & -l_2 \\ -l_3 & 0 & l_1 \\ l_2 & -l_1 & 0 \end{pmatrix}.$$

1.3 Matrix representation of infinitesimal rotations in relative coordinate system

For an infinitesimal rotation $\mathrm{d}\varphi$, $\sin(\mathrm{d}\varphi) \cong \mathrm{d}\varphi$, $\cos(\mathrm{d}\varphi) \cong 1$, and the rotation matrix becomes

$$\mathbf{R}_{(\mathrm{d}\varphi,\boldsymbol{l})} = \mathbf{E} + \mathrm{d}\mathbf{S}, \tag{4}$$

where $\mathrm{d}\mathbf{S} = -\mathrm{d}\varphi(\boldsymbol{\epsilon}\boldsymbol{l})$, and has inverse

$$\mathbf{R}^{-1}_{(\mathrm{d}\varphi,\boldsymbol{l})} = \mathbf{R}^{T}_{(\mathrm{d}\varphi,\boldsymbol{l})} = \mathbf{E} - \mathrm{d}\mathbf{S}. \tag{5}$$

Consider now the rotation matrix $\mathbf{Q}$ that corresponds to the composition of three infinitesimal rotations around axes $(\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3)$

$$\mathbf{Q} = \mathbf{R}_{(\mathrm{d}\varphi_1,\boldsymbol{l}_1)}\mathbf{R}_{(\mathrm{d}\varphi_2,\boldsymbol{l}_2)}\mathbf{R}_{(\mathrm{d}\varphi_3,\boldsymbol{l}_3)} = (\mathbf{E}+\mathrm{d}\mathbf{S}_1)(\mathbf{E}+\mathrm{d}\mathbf{S}_2)(\mathbf{E}+\mathrm{d}\mathbf{S}_3) \cong \mathbf{E}+\mathrm{d}\mathbf{S}_1+\mathrm{d}\mathbf{S}_2+\mathrm{d}\mathbf{S}_3. \tag{6}$$

For infinitesimal rotations $(\mathrm{d}\theta_1, \mathrm{d}\theta_2, \mathrm{d}\theta_3)$ around the axes of the relative coordinate system $(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$, the composite rotation matrix is

$$\mathbf{Q} = \mathbf{E} + \mathrm{d}\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{d}\theta_3 & \mathrm{d}\theta_2 \\ \mathrm{d}\theta_3 & 0 & -\mathrm{d}\theta_1 \\ -\mathrm{d}\theta_2 & \mathrm{d}\theta_1 & 0 \end{pmatrix}. \tag{7}$$

1.4 Change in the orientation of the relative coordinate system

During the infinitesimal time step dt the relative coordinate axes undergo a rotation

$$\mathbf{U}(t+\mathrm{d}t) = \mathbf{Q}\mathbf{U}(t) = (\mathbf{E}+\mathrm{d}\mathbf{S})\mathbf{U}(t),$$

which gives a differential system

$$\dot{\mathbf{U}} = \dot{\mathbf{S}}\,\mathbf{U}, \tag{8}$$

describing the change in the relative system orientation in terms of the rotation velocities around the axes of the relative system

$$\dot{\mathbf{S}}(\boldsymbol{\omega}) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}. \tag{9}$$

The current orientation of the relative coordinate system $\mathbf{U}(t)$ is the solution of the initial value problem (IVP)

$$\begin{array}{lcl} \dot{\boldsymbol{\omega}} & = & \boldsymbol{\varepsilon}, \\ \dot{\mathbf{U}} & = & \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}\mathbf{U}, \\ \mathbf{U}(0) & = & \mathbf{E}, \\ \boldsymbol{\omega}(0) & = & \mathbf{0}. \end{array}$$

1.5 Change in position of the relative coordinate system

The cell-phone measures accelerations $\boldsymbol{a}$ along axes of the relative coordinate system, that are expressed in the global coordinate system as

$$\ddot{\boldsymbol{X}} = \mathbf{U}\,\boldsymbol{a}.$$

The overall IVP to find the position is

$$\begin{array}{lcl} \dot{\boldsymbol{\omega}} & = & \boldsymbol{\varepsilon}, \\ \dot{\mathbf{U}} & = & \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}\mathbf{U}, \\ \dot{\boldsymbol{X}} & = & \boldsymbol{V}, \\ \dot{\boldsymbol{V}} & = & \mathbf{U}\,\boldsymbol{a}, \\ \boldsymbol{\omega}(0) & = & \mathbf{0}, \\ \mathbf{U}(0) & = & \mathbf{E}, \\ \boldsymbol{X}(0) & = & \mathbf{0}, \\ \boldsymbol{V}(0) & = & \mathbf{0}, \end{array} \tag{10}$$

where the linear, angular accelerations $\boldsymbol{a}, \boldsymbol{\varepsilon}$ are known from measurements.

2 Qualitative data analysis

A first step in processing the data $(\boldsymbol{a}, \boldsymbol{\varepsilon})$ acquired from the cell phone is to carry out some basic qualitative analysis, shown here using Octave (Matlab clone).

2.1 Data input

Cell-phone data saved in CSV format is read in and examined through the following steps:

1. Set the current directory

```
octave> dir='/home/student/courses/MATH590/NUMdata';
        chdir(dir);
octave>
```

2. Read the CSV-formatted data, and find the size of the data

```
octave> data=csvread('Mitran.csv');
        [m,d] = size(data); disp([m d]);
      18841        8
octave>
```

3. Visually examine some of the data

```
octave> data(1:10,:)
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.004 & -0.7803 & 0.2999 & -1.799 & -0.1402 & 0.2944 & 0.0533 & 0 \\ 0.005 & -0.7803 & 0.2999 & -1.799 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.005 & -0.0953 & 0.0762 & -0.0111 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.006 & -0.0953 & 0.0762 & -0.0111 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.006 & -0.0953 & 0.0762 & -0.0111 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.006 & -0.0953 & 0.0762 & -0.0111 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.006 & -0.0953 & 0.0762 & -0.0111 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.006 & -0.0953 & 0.0762 & -0.0111 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.007 & -0.0953 & 0.0762 & -0.0111 & -0.0003 & 0.7379 & -0.0279 & 0 \end{pmatrix}$$

```
octave>
```

4. Each line contains $(t_k, \boldsymbol{a}_k, \boldsymbol{\varepsilon}_k)$, with t_k the elapsed time, $\boldsymbol{a}_k$ the linear accelation at time t_k, $\boldsymbol{\varepsilon}_k$ the angular acceleration at time t_k. Note that there are several repeated t_k values. Since solving the IVP (10) requires definition of the functions $(\boldsymbol{a}(t), \boldsymbol{\varepsilon}(t))$, some initial processing is required to eliminate duplicate values. The Octave `unique` function is used to obtain a vector `t` of the `mu` unique values, and the indices `iu` at which each unique value is first encountered.

```
octave> [t,iu]=unique(data(:,1));
octave> t(1:10)'
```

(0 0.004 0.005 0.006 0.007 0.025 0.056 0.068 0.101 0.119)

```
octave> iu(1:16)'
```

(1 2 4 9 11 12 15 18 19 20 22 24 25 26 28 31)

```
octave> mu=max(size(iu))
```

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```
octave> t(mu-6:mu)'
```

(162.851 162.852 162.88 162.881 162.91 162.93 162.944)

```
octave> iu(mu-6:mu)'
```

(18830 18831 18832 18834 18837 18838 18841)

```
octave> raddeg=pi/180
```

0.017453

```
octave>
```

5. Define vectors `a`, `epsilon` containing unique linear, angular acceleration values.

```
octave> a=data(iu,2:4); epsilon=data(iu,5:7)/raddeg;
octave> [a(1:6,:) epsilon(1:6,:)]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -0.7803 & 0.2999 & -1.799 & -8.0329 & 16.868 & 3.0539 \\ -0.0953 & 0.0762 & -0.0111 & 0.085944 & 21.033 & 12.152 \\ -0.0953 & 0.0762 & -0.0111 & 0.085944 & 21.033 & 12.152 \\ -0.6242 & -0.0203 & -0.9742 & -0.017189 & 42.279 & -1.5986 \\ -0.6242 & -0.0203 & -0.9742 & -0.017189 & 42.279 & -1.5986 \end{pmatrix}$$

```
octave> data(1:10,:)
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.004 & -0.7803 & 0.2999 & -1.799 & -0.1402 & 0.2944 & 0.0533 & 0 \\ 0.005 & -0.7803 & 0.2999 & -1.799 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.005 & -0.0953 & 0.0762 & -0.0111 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.006 & -0.0953 & 0.0762 & -0.0111 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.006 & -0.0953 & 0.0762 & -0.0111 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.006 & -0.0953 & 0.0762 & -0.0111 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.006 & -0.0953 & 0.0762 & -0.0111 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.006 & -0.0953 & 0.0762 & -0.0111 & 0.0015 & 0.3671 & 0.2121 & 0 \\ 0.007 & -0.0953 & 0.0762 & -0.0111 & -0.0003 & 0.7379 & -0.0279 & 0 \end{pmatrix}$$

```
octave>
```

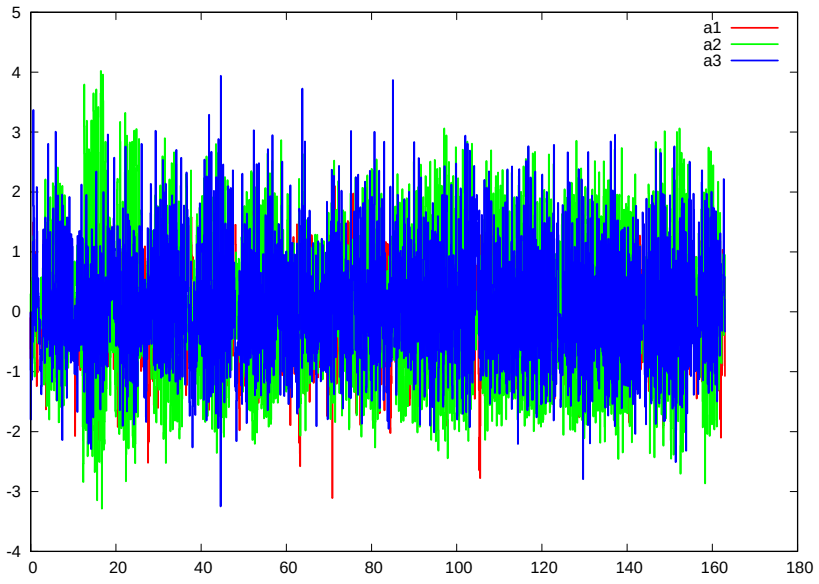
2.2 Data plotting

Having ensured data corresponds to functions $\boldsymbol{a}(t), \boldsymbol{\varepsilon}(t)$, a next step is data visualization. This shall be accomplished by writing the data to a text file and using the Gnuplot utility.

```
octave> fid=fopen('Mitran.data','w');
        fprintf(fid,'%f %f %f %f %f %f %f\n',[t a epsilon]');
        fclose(fid);
octave>
```

2.2.1 Linear accelerations

```
GNUplot] cd '/home/student/courses/MATH590/NUMdata'
         set terminal postscript eps enhanced color
         set style line 1 lt 2 lc rgb "red" lw 3
         set style line 2 lt 2 lc rgb "green" lw 3
         set style line 3 lt 2 lc rgb "blue" lw 3
         plot 'Mitran.data' u 1:2 w l ls 1 title "a1", '' u 1:3 w l ls 2
         title "a2", '' u 1:4 w l ls 3 title "a3"
```




```
GNUplot]
```

Observations on the above plot (qualitative data analysis):

- average accelerations seem to be zero as expected for a closed path. This can readily be checked

```
octave> mean(a)
```

(0.014903 0.11257 0.065759)

```
octave> std(a)
```

(0.68179 1.2446 0.97412)

```
octave> mean(a)./std(a)
```

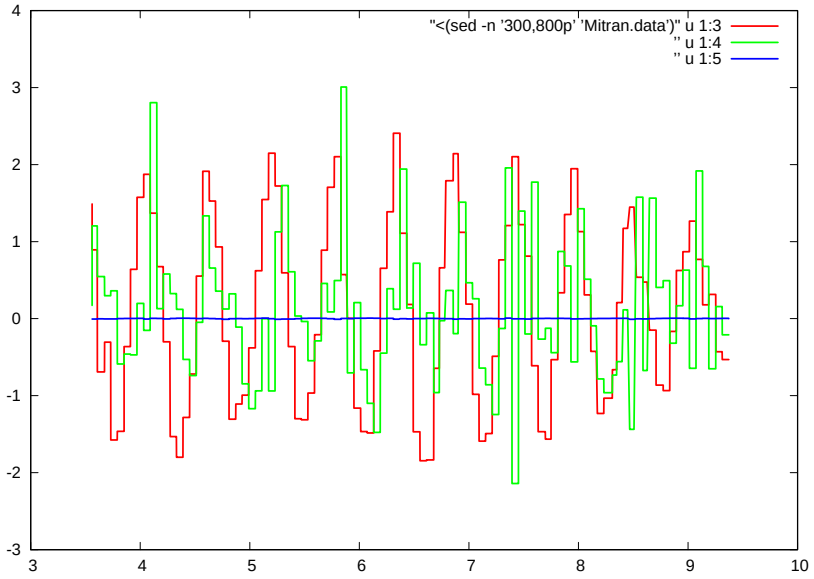
(0.021859 0.090447 0.067506)

```
octave>
```

From the above, the ratios of the means to the standard deviations are between 2% and 9%, hence relatively small.

In addition to the overall plot, it is useful to plot smaller time windows

```
GNUplot] cd '/home/student/courses/MATH590/NUMdata'
         set terminal postscript eps enhanced color
         set style line 1 lt 2 lc rgb "red" lw 3
         set style line 2 lt 2 lc rgb "green" lw 3
         set style line 3 lt 2 lc rgb "blue" lw 3
         plot "<(sed -n '300,800p' 'Mitran.data')" u 1:3 w l ls 1, '' u 1:4
         w l ls 2, '' u 1:5 w l ls 3
```

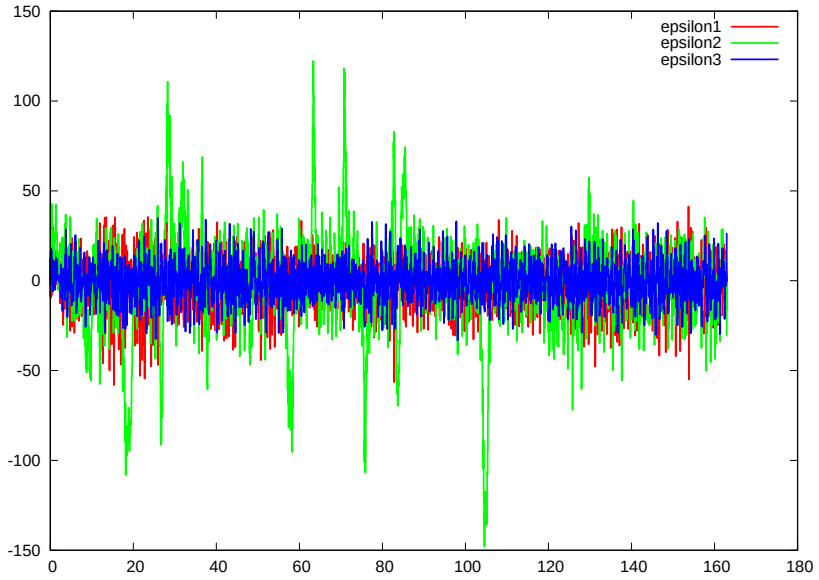



```
GNUplot]
```

It is remarkable that a clear waveform appears in the data. The wave form along $\boldsymbol{u}_1$ corresponds to left-right accelerations during bipedal walking. The wave form along $\boldsymbol{u}_2$ corresponds to up-down accelerations.

2.2.2 Angular accelerations

```
GNUplot] cd '/home/student/courses/MATH590/NUMdata'
         set terminal postscript eps enhanced color
         set style line 1 lt 2 lc rgb "red" lw 3
         set style line 2 lt 2 lc rgb "green" lw 3
         set style line 3 lt 2 lc rgb "blue" lw 3
         plot 'Mitran.data' u 1:5 w l ls 1 title "epsilon1", '' u 1:6 w l ls
         2 title "epsilon2", '' u 1:7 w l ls 3 title "epsilon3"
```




```
GNUplot]
```

Observations:

– There are bursts of large angular accelerations around the $\boldsymbol{u}_2$ axis. As will be seen later, this generally corresponds to the vertical axis and the bursts correspond to turns in direction of the walker.

3 Trajectory reconstruction

3.1 Integration

3.1.1 Reconstructing angular velocities from gyroscope measurements

Recall that the integration

$$\boldsymbol{\omega}(t) = \int_0^t \boldsymbol{\varepsilon}(\tau)\mathrm{d}\tau \Rightarrow \boldsymbol{\omega}_i = \boldsymbol{\omega}(t_i) = \sum_{j=1}^{i-1} h_j \boldsymbol{\varepsilon}_j, h_j = t_{j+1} - t_j$$

is essentially a non-compressive data transformation. The above is a cummulative sum, readily evaluated in Octave

```
octave> shift([1 2 3 4],-1)
```

$$\begin{pmatrix} 2 & 3 & 4 & 1 \end{pmatrix}$$

```
octave> omega=zeros([mu,3]);
        h = shift(t,-1)-t;
octave> [h(1:10) t(1:10)]
```

$$\begin{pmatrix} 0.004 & 0 \\ 0.001 & 0.004 \\ 0.001 & 0.005 \\ 0.001 & 0.006 \\ 0.018 & 0.007 \\ 0.031 & 0.025 \\ 0.012 & 0.056 \\ 0.033 & 0.068 \\ 0.018 & 0.101 \\ 0.001 & 0.119 \end{pmatrix}$$

```
octave> omega(2:mu,1) = cumsum(h(1:mu-1).*epsilon(1:mu-1,1));
        omega(2:mu,2) = cumsum(h(1:mu-1).*epsilon(1:mu-1,2));
        omega(2:mu,3) = cumsum(h(1:mu-1).*epsilon(1:mu-1,3));
octave> [omega(1:6,1) epsilon(1:6,1)]
```

$$\begin{pmatrix} 0 & 0 \\ 0 & -8.0329 \\ -0.0080329 & 0.085944 \\ -0.0079469 & 0.085944 \\ -0.007861 & -0.017189 \\ -0.0081704 & -0.017189 \end{pmatrix}$$

```
octave>
```

3.1.2 Reconstructing the relative coordinate system basis

The same procedure can be used to compute the time history of the relative reference frame orientation $\mathbf{U}(t)$

$$\dot{\mathbf{U}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \mathbf{U} = \mathbf{S}\mathbf{U} \Rightarrow \mathbf{U}(t_i) = \sum_{j=1}^{i-1} h_j \, \mathbf{S}_j \, \mathbf{U}_j$$

```
octave> U=zeros([mu,3,3]); I3=eye(3); U(1,:,:)=I3; U0=I3;
```

```
octave> i=1; while(i<mu)
          om(1:3) = omega(i,1:3);
          S=[0 -om(3) om(2); om(3) 0 -om(1); -om(2) om(1) 0];
          [U1,R1] = qr((I3 + h(i)*S)*U0);
          U1 = U1*diag(sign(diag(R1)));
          U(i+1,:,:)=U1;
          i++; U0=U1;
        endwhile
```

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```
octave>
```

Display of successive $\mathbf{U}(t_i)$ clearly shows the rotation of the axes carried by the cell phone

```
octave> i=1; di=50; while(i<300)
          U1=reshape(U(i,:,:),3,3);
          disp(strcat('At time t=',num2str(t(i)))); disp(U1); disp(' ');
          i=i+di;
        endwhile
   At time t=0
      1   0   0
      0   1   0
      0   0   1

   At time t=0.671
      0.815161  -0.376014   0.440596
      0.080612   0.826896   0.556548
     -0.573597  -0.418159   0.704365

   At time t=1.266
      0.110337  -0.982359   0.150984
      0.200490   0.170788   0.964694
     -0.973463  -0.076171   0.215797

   At time t=1.858
      0.675025  -0.721612   0.153679
      0.049460   0.252085   0.966440
     -0.736135  -0.644770   0.205855

   At time t=2.502
     -0.96946  -0.20458   0.13529
      0.16788  -0.15139   0.97411
     -0.17880   0.96707   0.18111

   At time t=3.062
     -0.385709   0.914210   0.124289
```

```
   -0.080865  -0.167693   0.982517
    0.919070   0.368915   0.138608
```

1

```
octave>
```

3.1.3 Reconstructing the walker velocity and position

Integration of $\dot{\boldsymbol{V}} = \mathbf{U}\boldsymbol{a}$ and $\dot{\boldsymbol{X}} = \boldsymbol{V}$ provides the walker trajectory

```
octave> V=zeros([mu,3]); X=V; I3=eye(3); U(1,:,:)=I3; U0=I3;
octave> i=1; while(i<mu)
          U1=reshape(U(i+1,:,:),3,3);
          V(i+1,:) = V(i,:) + h(i)*a(i+1,:)*U1';
          X(i+1,:) = X(i,:) + h(i)*V(i+1,:);
          i++;
        endwhile
```

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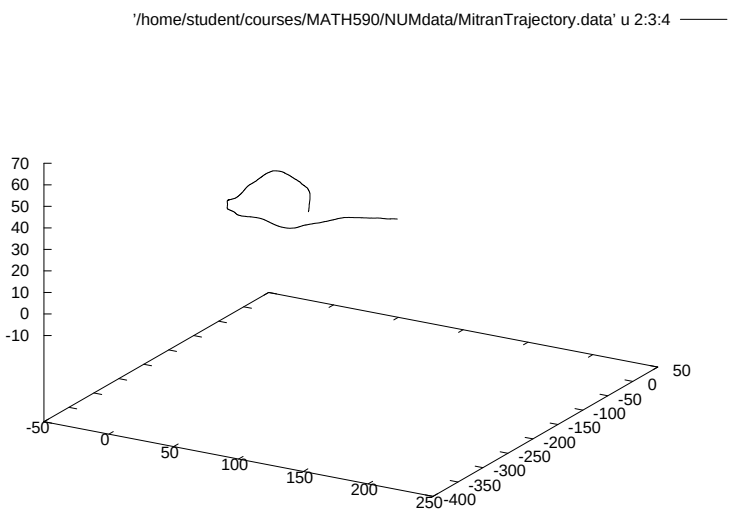
```
octave>
```

Write the resulting positions and velocities to a file in preparation for plotting

```
octave> fid=fopen('MitranTrajectory.data','w');
        fprintf(fid,'%f %f %f %f %f %f %f\n',[t X V]');
        fclose(fid);
octave>
```

Plot the trajectory

```
GNUplot] splot '/home/student/courses/MATH590/NUMdata/MitranTrajectory.data'
         u 2:3:4 w l
```



```
GNUplot]
```

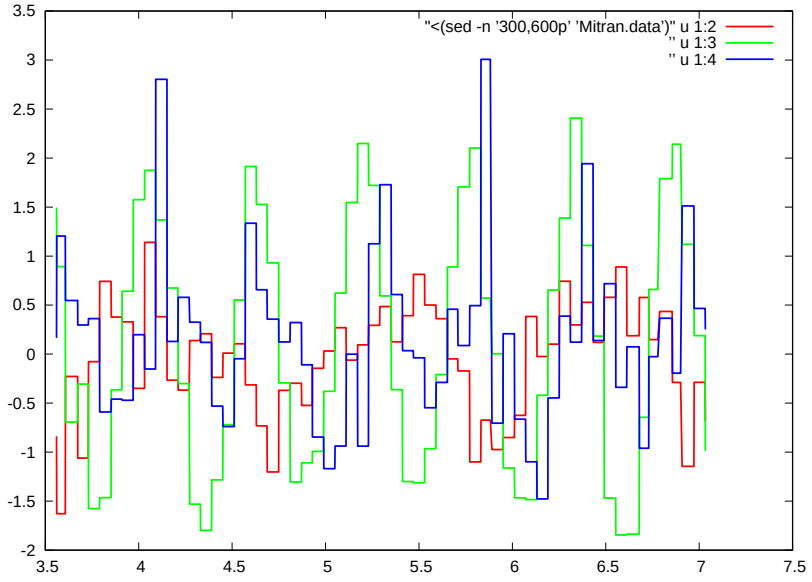
4 Trajectory analysis

After reconstructing the trajectory from accelerometer data, the problem of extracting individual walker features from the data is considered.

4.1 Interpolation

Recall that a regular pattern is apparent in the accelerometer data, one that we associate with the walker's gait. We now consider the problem of characterizing the gait

```
GNUplot] cd '/home/student/courses/MATH590/NUMdata'
         set terminal postscript eps enhanced color
         set style line 1 lt 2 lc rgb "red" lw 3
         set style line 2 lt 2 lc rgb "green" lw 3
         set style line 3 lt 2 lc rgb "blue" lw 3
         plot "<(sed -n '300,600p' 'Mitran.data')" u 1:2 w l ls 1, '' u 1:3
         w l ls 2, '' u 1:4 w l ls 3
```




```
GNUplot]
```

1. Extract a data window of interest

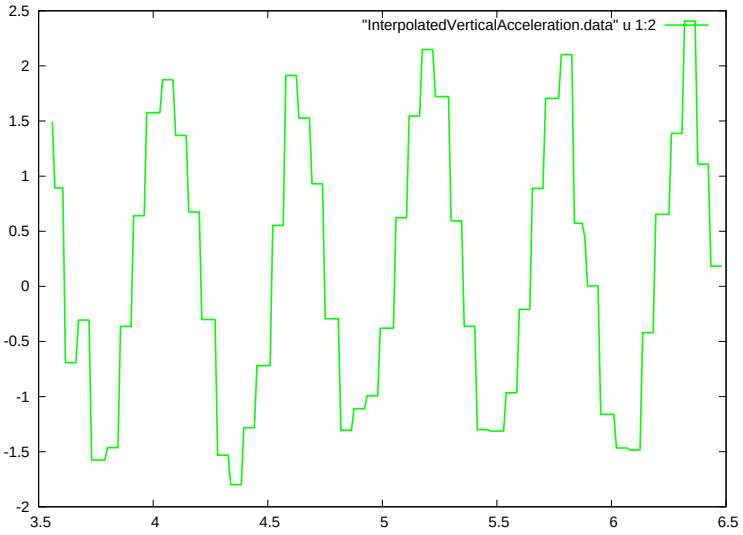
   ```
   octave> iG0=300; nG=256; iG1=iG0+nG-1;
   octave> tG=t(iG0:iG1); aG=a(iG0:iG1,:);
   octave>
   ```

2. Interpolate to obtain data at constant-spaced time increments. The second component of the acceleration vector, corresponding to vertical motion is chosen (green line in above plot).

```
octave> tG0=tG(1); tG1=tG(nG); dt=(tG1-tG0)/nG;
octave> tGi=tG0+(0:nG-1)*dt;
octave> aGi=interp1(tG',aG(:,2)',tGi);
octave> fid=fopen('InterpolatedVerticalAcceleration.data','w');
        ta = [tGi' aGi'];
        fprintf(fid,'%f %f\n',ta');
        fclose(fid);
octave>
```

Plot the interpolated data

```
GNUplot] cd '/home/student/courses/MATH590/NUMdata'
         set terminal postscript eps enhanced color
         set style line 1 lt 2 lc rgb "green" lw 3
         plot "InterpolatedVerticalAcceleration.data" u 1:2 w l ls 1
```



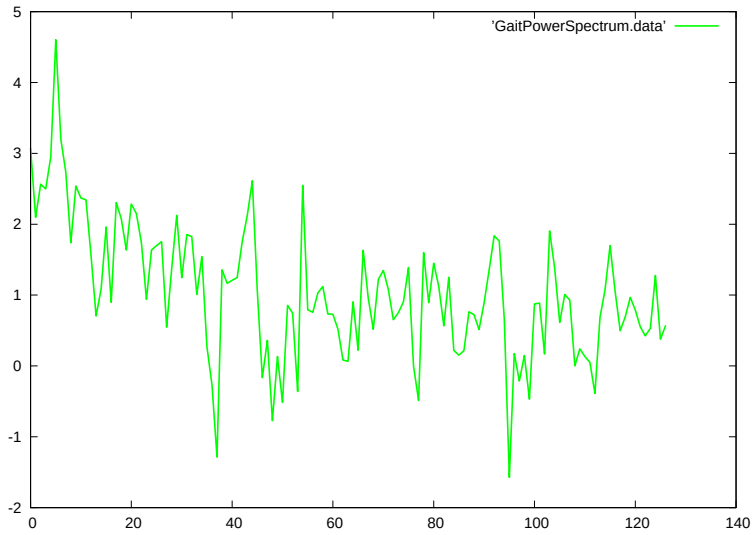

```
GNUplot]
```

3. Find the period by Fourier analysis

```
octave> AGi=fft(aGi); PAGi = log10(AGi.*conj(AGi));
octave> fid=fopen('GaitPowerSpectrum.data','w');
        fprintf(fid,'%f\n',PAGi(1:nG/2-1));
        fclose(fid);
octave>
```

A plot of the power spectrum (in logarithmic coordinates) shows a distinct maximum

```
GNUplot] cd '/home/student/courses/MATH590/NUMdata'
         set terminal postscript eps enhanced color
         set style line 1 lt 2 lc rgb "green" lw 3
         plot 'GaitPowerSpectrum.data' w l ls 1
```




```
GNUplot]
```

```
octave> [val,idx] = max(PAGi); disp([val idx]);
      4.6046    6.0000
octave>
```

4. Determine the period

```
octave> TG = (tG1-tG0)/(idx-1)
   0.5862
octave> nT=floor(max(size(aGi))/(idx-1))
   51
octave>
```

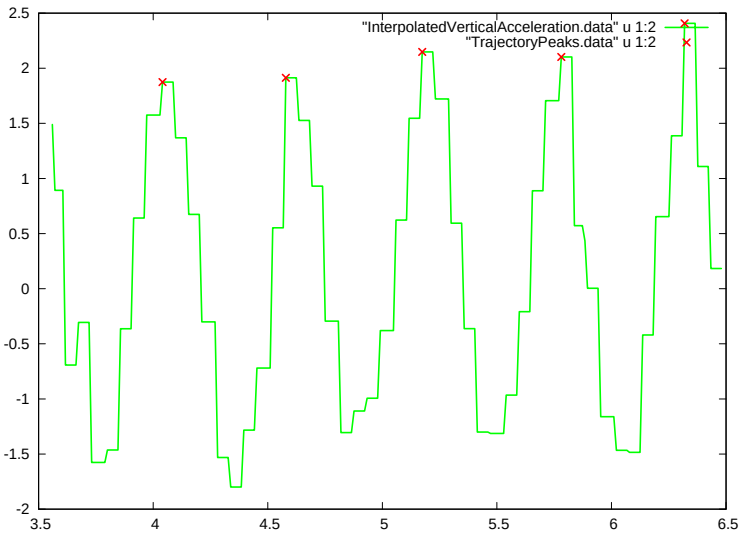
5. Determine locations of the peak vertical accelerations. Note that the particular values for `MinPeakHeight, MinPeakDistance` have to be determined by experimentation, checking the plot to see the peaks have been identified.

```
octave> [aPeak, iPeak] = findpeaks(aGi-min(aGi),"MinPeakHeight",1.,
        "MinPeakDistance",nT/3);
        fid=fopen('TrajectoryPeaks.data','w');
        ta = [tGi(iPeak)' (aPeak+min(aGi))'];
        fprintf(fid,'%f %f\n',ta');
        fclose(fid);
octave> nPeriods = max(size(iPeak))-1
   4
octave> iPeak
   ( 43  90  142  195  242 )
octave> shift(iPeak,-1)
```

(90 142 195 242 43)

```
octave>
```

```
GNUplot] cd '/home/student/courses/MATH590/NUMdata'
         set terminal postscript eps enhanced color
         set style line 1 lt 2 lc rgb "red" lw 3
         set style line 2 lt 2 lc rgb "green" lw 3
         set style line 3 lt 2 lc rgb "blue" lw 3
         plot "InterpolatedVerticalAcceleration.data" u 1:2 w l ls 2,
         "TrajectoryPeaks.data" u 1:2 w p ls 1
```




```
GNUplot]
```

6. Extract data during `nPeriods`

   a) First find the maximum number of data points between the maxima, allocate space for the data, and extract the measured cycles

```
octave> nT = (shift(iPeak,-1)-iPeak+1)(1:nPeriods)
```

(48 53 54 48)

```
octave> nTmax = max(nT);
        tT = zeros([nTmax,nPeriods]);
        aT = zeros([nTmax,nPeriods]);
        TT = zeros([nPeriods,1]);
        i=1; while(i<=nPeriods)
          tT(1:nT(i),i) = tGi(iPeak(i):iPeak(i+1));
          aT(1:nT(i),i) = aGi(iPeak(i):iPeak(i+1));
          TT(i) = tT(nT(i),i) - tT(1,i);
          i++;
        endwhile;
        disp(TT');
```

```
      0.53811   0.59536   0.60681   0.53811
octave>
```

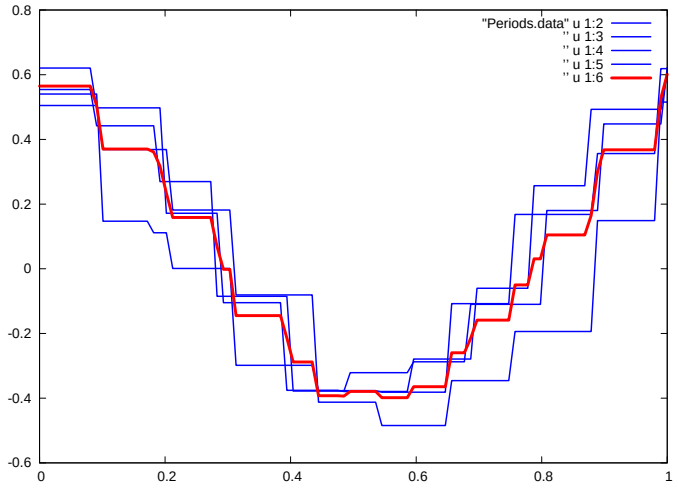
b) Scale the data to have the same amplitude and period

```
octave> i=1; while(i<=nPeriods)
          tT(1:nT(i),i) = tT(1:nT(i),i) - tT(1,i);
          tT(1:nT(i),i) = tT(1:nT(i),i)/TT(i);
          amp = max(aT(1:nT(i),i)) - min(aT(1:nT(i),i));
          aT(1:nT(i),i) = aT(1:nT(i),i)/amp;
          i++;
        endwhile;
octave>
```

c) Interpolate to obtain periods with equal number of samples, and also define an average waveform **g** over the periods

```
octave> nTi=100;
        dt=1./(nTi-1); ti=(0:nTi-1)*dt; ti=ti';
        aTi=zeros([nTi,nPeriods]); g=zeros([nTi,1]);
        i=1; while(i<=nPeriods)
          ai = interp1(tT(1:nT(i),i),aT(1:nT(i),i),ti',
        "nearest");
          aTi(:,i) = ai;
          g = g + ai';
          i++;
        endwhile;
        g = g/nPeriods;
        g = g/(max(g)-min(g));
octave> fid=fopen('Periods.data','w');
        ta = [ti aTi g];
        fprintf(fid,'%f %f %f %f %f %f\n',ta');
        fclose(fid);
        fid=fopen('AveragePeriod.data','w');
        ta = [ti g];
        fprintf(fid,'%f %f \n',ta');
        fclose(fid);
octave>

GNUplot] cd '/home/student/courses/MATH590/NUMdata'
         set terminal postscript eps enhanced color
         set style line 1 lt 2 lc rgb "red" lw 6
         set style line 2 lt 2 lc rgb "green" lw 3
         set style line 3 lt 2 lc rgb "blue" lw 3
         plot "Periods.data" u 1:2 w l ls 3, '' u 1:3 w l ls 3,
         '' u 1:4 w l ls 3, '' u 1:5 w l ls 3, '' u 1:6 w l ls 1
```



GNUplot]

4.2 Least squares

Seek a more economical representation of the average gait waveform, currently stored as a vector $\mathbf{g} \in \mathbb{R}^m$ of the vertical acceleration values at times within the vector $\mathbf{t} \in \mathbb{R}^m$. For example, consider approximating the waveform by a parabola $p(t) = c_0 + c_1 t + c_2 t^2$, leading to the least squares problem

$$\min_{\mathbf{c}\in\mathbb{R}^3} \|\mathbf{L}\mathbf{c} - \mathbf{g}\|, \mathbf{L} = \begin{pmatrix} \mathbf{1} & \mathbf{t} & \mathbf{t}^2 \end{pmatrix} \in \mathbb{R}^{m\times 3}, \mathbf{t}^k = \begin{pmatrix} t_1^k & \dots & t_m^k \end{pmatrix}^T. \tag{11}$$

A solution is found by projection onto the column space of $\mathbf{L}$,

$$\mathbf{QR} = \mathbf{L}, \mathbf{P}_{C(\mathbf{L})} = \mathbf{QQ}^T, \mathbf{P}_{C(\mathbf{L})}\mathbf{g} = \mathbf{QRc} \Rightarrow \mathbf{Rc} = \mathbf{Q}^T\mathbf{g}. \tag{12}$$

```
octave> L=[ti.^0 ti ti.^2]; [Q,R]=qr(L,0); [size(Q); size(R)]
```

$$\begin{pmatrix} 100 & 3 \\ 3 & 3 \end{pmatrix}$$

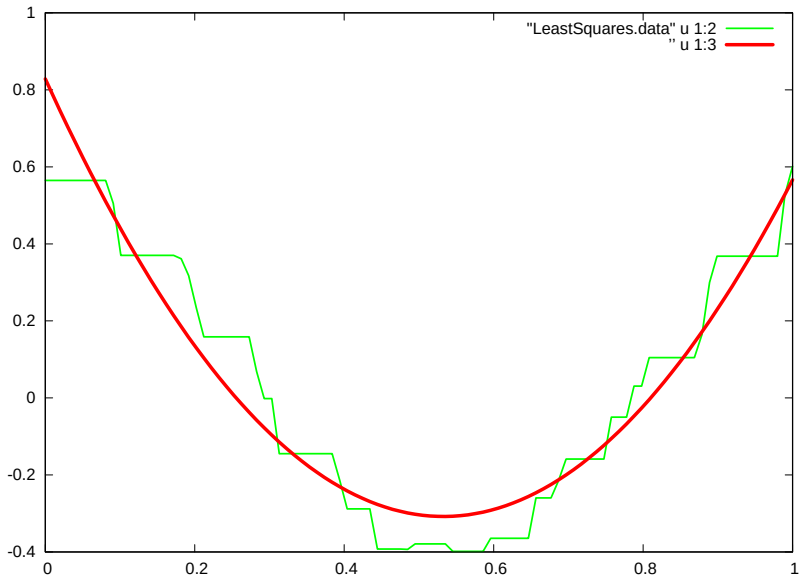
```
octave> c = R \ Q'*g
```

$$\begin{pmatrix} 0.82834 \\ -4.2652 \\ 4.003 \end{pmatrix}$$

```
octave> p = L*c;
        fid=fopen('LeastSquares.data','w');
        ta = [ti g p];
        fprintf(fid,'%f %f %f\n',ta');
        fclose(fid);
```

```
octave>
```

```
GNUplot] cd '/home/student/courses/MATH590/NUMdata'
         set terminal postscript eps enhanced color
         set style line 1 lt 2 lc rgb "red" lw 6
         set style line 2 lt 2 lc rgb "green" lw 3
         set style line 3 lt 2 lc rgb "blue" lw 3
         plot "LeastSquares.data" u 1:2 w l ls 2, '' u 1:3 w l ls 1
```




```
GNUplot]
```

4.3 Min-max

An alternative representation is through a linear combination of Chebyshev polynomials,

$$q(t) = d_0 T_0(t) + d_1 T_1(t) + d_2 T_2(t)$$

known to be a good approximation of the min-max polynomial of the data. The coefficient vector $\mathbf{d} \in \mathbb{R}^3$ is more difficult to find by comparison to the least squares case, and is carried out through a procedure known as the exchange algorithm, implemented in Octave by the `polyfitinf` function.

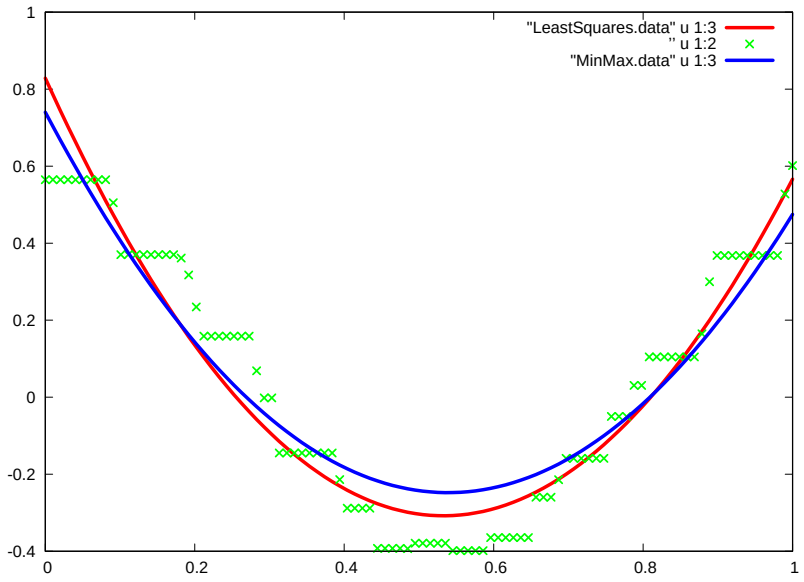
```
octave> d=polyfitinf(2,nTi,0,ti,g,5.0E-7,nTi)
```

$$\begin{pmatrix} 3.3996 & -3.665 & 0.74003 \end{pmatrix}$$

```
octave> q = polyval(d,ti);
        fid=fopen('MinMax.data','w');
        ta = [ti g q];
        fprintf(fid,'%f %f %f\n',ta');
        fclose(fid);
```

```
octave>
```

```
GNUplot] cd '/home/student/courses/MATH590/NUMdata'
         set terminal postscript eps enhanced color
         set style line 1 lt 2 lc rgb "red" lw 6
         set style line 2 lt 2 lc rgb "green" lw 3
         set style line 3 lt 2 lc rgb "blue" lw 6
         plot "LeastSquares.data" u 1:3 w l ls 1, '' u 1:2 w p ls 2,
         "MinMax.data" u 1:3 w l ls 3
```




```
GNUplot]
```