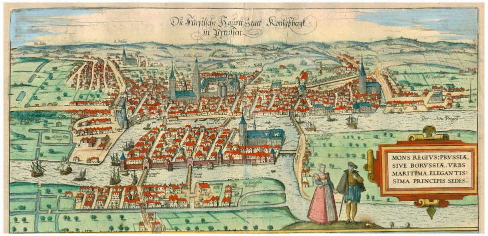
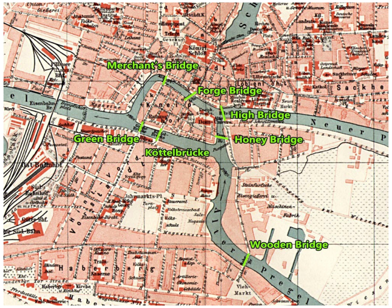
Module overview

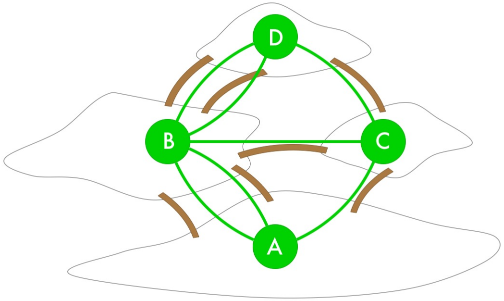
Topology is a branch of mathematics that studies what properties of an object are preserved under continuous deformation. Geometry studies equivalence under translation (congruence), uniform scaling (similarity), or projection. Topology extends this to study of all continuous transformations, such that a circle becomes equivalent to a square. Concepts from topology are relevant to data analysis, in particular in ascertaining efficient representations of sparse data sets.

- Brief history: Bridges of Königsberg, Hairy ball theorem
- Formal definition in terms of set theory
- Complexes
- Morse functions
- Persistent homology

- Königsberg (Kaliningrad) Mayor writes to Leonhard Euler in 1735 to ask: is there a way to cross all bridges in his city only once? Euler complains that the question is not within the sphere of interest of a mathematician.
- Solution did not depend on "geometry" (i.e., measurements of a space), but only on *connectivity* leading to both topology and graph theory







- Further thought by Euler led to his polyhedron formula (1750): $V - E + F = 2$
- Notice that the structure of the graph of the Königsberg bridges is markedly different from the structure of the real line or real plane

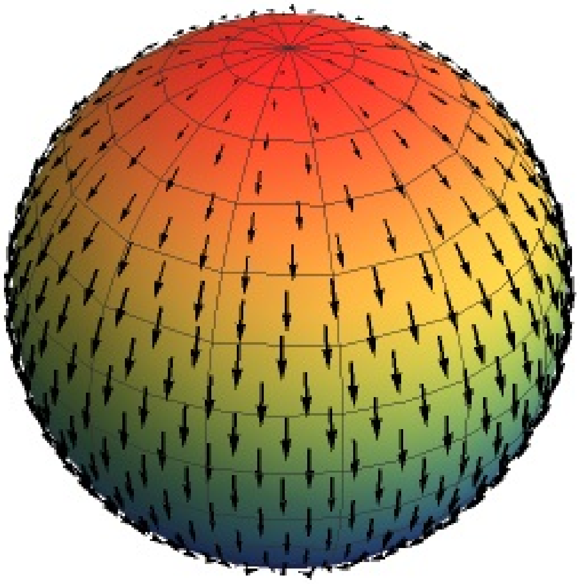
History of topology: Hairy ball theorem

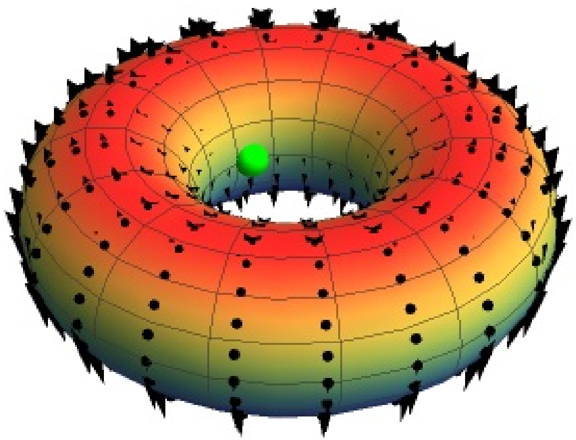
- There exists no non-vanishing continuous tangent vector field on even-dimensional spheres, or, ...

  "you can't comb a hairy ball flat without creating a cowlick"

  "you can't comb the hair on a coconut"

  Every smooth vector field on a sphere has a singular point





Definition. *A* topological space *is an ordered pair* $(\mathcal{A}, \mathcal{T} \subseteq 2^{\mathcal{A}})$ *of a set* $\mathcal{A}$ *and a set of subsets of* $\mathcal{A}$, $\mathcal{T}$ *called the* topology *on* $\mathcal{A}$ *such that:*

1. $\varnothing \in \mathcal{T}, \mathcal{A} \in \mathcal{T}$, *the empty set and the full set belong to the topology*
2. *Any union of elements of* $\mathcal{T}$ *is an element of* $\mathcal{T}$

$$\bigcup_{i \in I} \mathcal{S}_i \in \mathcal{T} \text{ if } \mathcal{S}_i \in \mathcal{T}$$

3. *Any finite intersection of elements of* $\mathcal{T}$ *is an element of* $\mathcal{T}$

$$\bigcap_{i=1}^{n} \mathcal{S}_i \in \mathcal{T}, i, n \in \mathbb{N}$$

Example. The *trivial topology* on $\mathcal{A}$ is $\mathcal{T} = \{\varnothing, \mathcal{T}\}$

A point $\boldsymbol{x} = \sum_{i=0}^{k} \lambda_i \boldsymbol{u}_i$, with $\lambda_i \in \mathbb{R}$ is an *affine combination* of points $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \in \mathbb{R}^d$, if $\sum_{i=0}^{k} \lambda_i = 1$.

The set of all affine combinations is the *affine hull* of points $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \in \mathbb{R}^d$.

$\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \in \mathbb{R}^d$ are *affinely independent* if any two affine combinations $\boldsymbol{x} = \sum_{i=0}^{k} \lambda_i \boldsymbol{u}_i$, $\boldsymbol{y} = \sum_{i=0}^{k} \mu_i \boldsymbol{u}_i$ are equal iff $\lambda_i = \mu_i$

An affine hull is a k-*plane* if $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \in \mathbb{R}^d$ are affinely independent.

The $k+1$ points $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k \in \mathbb{R}^d$ are affinely independent iff the vectors $\boldsymbol{u}_1 - \boldsymbol{u}_0, \ldots, \boldsymbol{u}_k - \boldsymbol{u}_0$ are linearly independent.

An affine combination $\boldsymbol{x} = \sum_{i=0}^{k} \lambda_i \boldsymbol{u}_i$ is a *convex combination* if $\lambda_i \geqslant 0$.

The set set of all convex combinations is a *convex hull*.

A k-*simplex* is convex hull of $k+1$ affinely independent points, $\sigma = \operatorname{conv}\{\boldsymbol{u}_0, \ldots, \boldsymbol{u}_k\}$, with dimension $\dim \sigma = k$

The 0,1,2,3-dimensional simplices are named: *vertex*, *edge*, *triangle*, *tetrahedron*.

Any subset of affinely independent points again defines a simplex.

A *face* τ of the simplex $\sigma = \mathrm{conv}\{\boldsymbol{u}_0, \ldots, \boldsymbol{u}_k\}$ is the convex hull of some subset of the points, $\tau \leqslant \sigma$. It is a proper face if the subset is not the entire set of points, $\tau < \sigma$, and σ is said to be the *coface* of τ.

The union of all proper faces is the *boundary* of σ, denoted as $\mathrm{bd}(\sigma)$. The interior is the complement of the boundary, $\mathrm{int}\,(\sigma) = \sigma - \mathrm{bd}(\sigma)$. A point $\boldsymbol{x} = \sum_{i=0}^{k} \lambda_i \boldsymbol{u}_i \in \sigma$ is in the interior, $\boldsymbol{x} \in \mathrm{int}(\sigma)$ if $\lambda_i > 0$. The point $\boldsymbol{x}$ belongs to the interior of the face spanned by the points for which $\lambda_i > 0$.

Definition. *A simplicial complex is a finite collection $\mathcal{K}$ of simplices such that $\sigma \in \mathcal{K}$ and $\tau \leqslant \mathcal{K}$ implies $\tau \in \mathcal{K}$, and $\sigma, \sigma_0 \in \mathcal{K}$ implies $\sigma \cap \sigma_0$ is either empty or a face of both.*

Restated, $\mathcal{K}$ is closed under taking faces and has no improper intersections.

The *dimension of simplicial complex* $\mathcal{K}$ is the maximum of any of its simplices.

The *underlying space* $|\mathcal{K}|$ is the union of its simplices together with the topology of the embedding Euclidean space ($\mathbb{R}^d$).

A *homeomorphism* between topological spaces $\mathcal{S}, \mathcal{T}$ is a function $f\colon \mathcal{S} \to \mathcal{T}$ that is a bijection, continuous and with continuous inverse.

A *triangulation* $\mathbb{T} = (\mathcal{K}, f\colon \mathcal{T} \to |\mathcal{K}|)$ of a topological space $\mathcal{T}$ is a simplicial complex $\mathcal{K}$ together with a homemorphism f.

A *subcomplex* of $\mathcal{K}$ is a complex $\mathcal{L}$ with $\mathcal{L} \subseteq \mathcal{K}$. The subcomplex is said to be *full* if it contains all simplices spanned by the vertices in $\mathcal{L}$.

The j-*skeleton* is the set of all simplices of dimension j or less, $\mathcal{K}^{(j)} = \{\sigma \in \mathcal{K} | \dim \sigma \leqslant j\}$. The 0-skeleton is the vertex set, $\mathrm{Vert}(\mathcal{K}) = \mathcal{K}^{(0)}$.

The star of simplex τ is the set of all cofaces, $\mathrm{St}(\tau) = \{\sigma \in \mathcal{K} | \tau \leqslant \sigma\}$. Generally the star is not closed under taking faces.

Consider a finite set of points $\mathcal{S} \subseteq \mathbb{R}^d$.

The *Voronoi cell* of a point $\boldsymbol{u} \in \mathcal{S}$ is $\mathcal{V}_{\boldsymbol{u}} = \{\boldsymbol{x} \in \mathbb{R}^d, \|\boldsymbol{x} - \boldsymbol{u}\| \leqslant \|\boldsymbol{x} - \boldsymbol{v}\|, \boldsymbol{v} \in \mathcal{S}\}$.

$\mathcal{V}_{\boldsymbol{u}}$ is a convex polyhedron in $\mathbb{R}^d$.

The *Voronoi diagram* of $\{\boldsymbol{u}_0, \ldots, \boldsymbol{u}_k\}$ is the collection of all the Voronoi cells $\{\mathcal{V}_{\boldsymbol{u}_0}, \ldots, \mathcal{V}_{\boldsymbol{u}_k}\}$

The *Delaunay complex* of $\mathcal{S} \subseteq \mathbb{R}^d$ is $\{\sigma \subseteq \mathcal{S} \,|\, \bigcap_{\boldsymbol{u} \in \sigma} \mathcal{V}_{\boldsymbol{u}} \neq \emptyset\}$.

The *Delaunay triangulation* is obtained by choosing σ to be convex hulls

A *homeomorphism* f is continuous function with continuous inverse between two topological spaces $(\mathcal{A}, \mathcal{T} \subseteq 2^{\mathcal{A}})$, $(\mathcal{B}, \mathcal{U} \subseteq 2^{\mathcal{A}})$, $f\colon \mathcal{A} \to \mathcal{B}$, $f \in C(\mathcal{A})$, $f^{-1} \in \mathcal{C}(\mathcal{B})$.

A *topological invariant* of the topological space $(\mathcal{A}, \mathcal{T} \subseteq 2^{\mathcal{A}})$ is a property that remains invariant under homeomorphism. Examples:

- Cardinality of $\mathcal{A}$, $|\mathcal{A}|$
- Cardinality of the topology on $\mathcal{A}$, $|\mathcal{T}|$

An *abelian group* is an algebraic structure that satisfies properties of: closure, associativity, existence of an indentity element, existence of inverse element, and commutativity

It is common to describe a topology by associating the topological space to a sequence of abelian groups, a procedure known as a *homology*. The *homology of a topological space* $(\mathcal{A}, \mathcal{T} \subseteq 2^{\mathcal{A}})$ is formally the set of topological invariants given by the homology groups $H_0(\mathcal{T}), H_1(\mathcal{T}), H_2(\mathcal{T}), \ldots$ that, informally characterize the holes in a manifold.

Given a topological space $(\mathcal{A}, \mathcal{T} \subseteq 2^{\mathcal{A}})$, the k^{th} Betti number $b_k(\mathcal{T})$ is the number of k-dimensional holes, e.g.

- b_0 is the number of connected components
- b_1 is the number of one-dimensional holes

The *free rank* of a group is the rank of the torsion-free part.

Formally, b_k is the the free rank of the singular homology group $H_k(\mathcal{T}, \mathbb{Z})$