APPROXIMATION TECHNIQUES - EXERCISES

1. Rate and order of convergence

Exercise 1. Estimate the rate and order of convergence of a sequence from the samples

$$1, \frac{0.9}{2}, \frac{0.81}{4}, \frac{0.729}{8}.$$

Exercise 2. Construct a convergence plot for the sequence

$$a_n = \frac{1}{\ln n}.$$

Can you estimate the rate and order of convergence?

Solution. Since $\lim_{n\to\infty} a_n = a = 0$, note that the distance to the limit $d_n = |a_n - a|$ is simply $d_n = a_n$ for $n > 1$. For large n, the convergence behavior is given by $d_{n+1} = s d_n^q$, leading to $\lg(d_{n+1}) = q \lg(d_n) + \lg(s)$ upon taking decimal logarithms. In practical computation, n is chosen between some upper bound N dictated by allowed computational effort and a lower bound M needed to start observing convergence

$$M \leqslant n \leqslant N.$$

The computed terms $\{d_n\}$ allow estimation of (s, q) by linear regression (least squares fitting) expressed as

$$\begin{bmatrix} 1 & \lg(d_M) \\ 1 & \lg(d_{M+1}) \\ \vdots & \vdots \\ 1 & \lg(d_{N-1}) \end{bmatrix} \begin{bmatrix} \lg(s) \\ q \end{bmatrix} \cong \begin{bmatrix} \lg(d_{M+1}) \\ \lg(d_{M+2}) \\ \vdots \\ \lg(d_N) \end{bmatrix} \Leftrightarrow \boldsymbol{A}\boldsymbol{x} \cong \boldsymbol{y}.$$

The Julia instruction $x = A \backslash y$ carries out the linear regression (also a feature of Matlab/Octave).

```
∴ M=20; N=100; n=M:N; a=1 ./ log.(n); d=a; lgd=log10.(d);
∴ m=N-M; y=lgd[2:m+1]; A=ones(m,2); A[:,2]=lgd[1:m];
∴ x=A\y; s=10^x[1]; q=x[2]; [s q]
```

$$[\ 0.9560115307957512 \quad 0.9712804466056952\ ] \tag{1}$$

```
∴
```

From the above, obtain order of convergence $q \cong 0.97$ (sublinear convergence), and rate of convergence $s \cong 0.96$. The convergence plot highlights this sublinear convergence behavior.

```
∴ x=n; y=d; clf(); semilogy(x,y,"."); grid("on");
∴ xlabel(L"$n$"); ylabel(L"$d_n$"); title("Convergence␣plot");
∴ savefig(homedir()*"/courses/MATH661/images/E02Fig01.eps");
∴
```

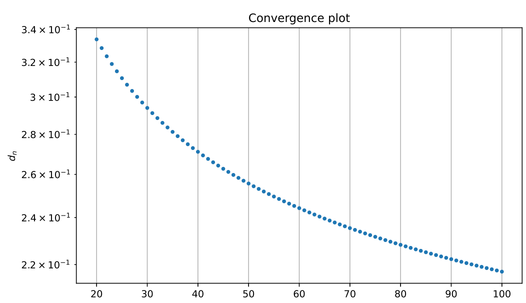

Figure 1. Sub-linear convergence behavior of the sequence $a_n = 1/\log(n)$.

Exercise 3. Construct a convergence plot for the sequence

$$a_n = e^{-n}.$$

Can you estimate the rate and order of convergence?

Solution. Again, $\lim_{n\to\infty} a_n = 0$, hence $d_n = a_n$ for $n \geqslant 1$. Repeat the above computation for the new sequence choosing lower values for M, N since e^{-n} is expected to converge to zero faster than $1/\ln n$.

```
∴ M=2; N=20; n=M:N; a=exp.(-n); d=a; lgd=log10.(d);
∴ m=N-M; y=lgd[2:m+1]; A=ones(m,2); A[:,2]=lgd[1:m];
∴ x=A\y; s=10^x[1]; q=x[2]; [s q]
```

$$[\ 0.36787944117144333 \quad 0.9999999999999999\ ] \tag{2}$$

Linear convergence is obtained, $q = 1$, and the rate of convergence $s \cong 0.37$. The order of convergence is only slightly better than the previous sequence and faster convergence arise from the marked difference in rate of convergence.

```
∴ x=n; y=d; clf(); semilogy(x,y,"."); grid("on");
```
```
∴ xlabel(L"$n$"); ylabel(L"$d_n$"); title("Convergence␣plot");
```
```
∴ savefig(homedir()*"/courses/MATH661/images/E02Fig02.eps");
```
```
∴
```

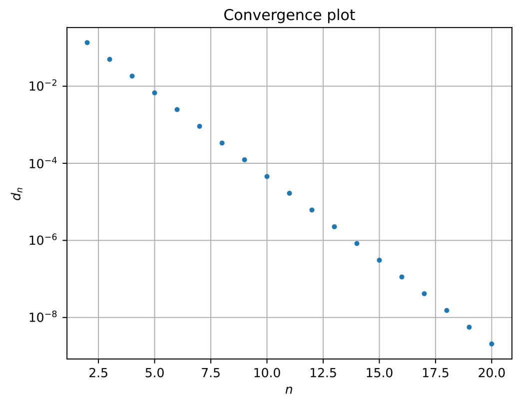

Figure 2. Sub-linear convergence behavior of the sequence $a_n = e^{-n}$.

Exercise 4. Plot in logarithmic coordinates $a_n = \sqrt{n}$, $a_{n+1} - a_n$, $a_{n+2} - a_n$, $\ldots$, $a_{n+10} - a_n$. Do the $a_{n+k} - a_n$ sequences converge?

2. Aitken acceleration

Exercise 5. What is the order of convergence of the sequence $x_n = 1/n$? Construct a convergence plot.

Solution. Proceed as above with $\lim_{n\to\infty} x_n = 0$, $d_n = x_n$ for $n \geqslant 1$. Repeat the above computations, changing notation for the new problem.

```
∴ M=20; N=100; n=M:N; x=1 ./ n; d=x; lgd=log10.(d);
```
```
∴ m=N-M; y=lgd[2:m+1]; A=ones(m,2); A[:,2]=lgd[1:m];
```
```
∴ b=A\y; s=10^b[1]; q=b[2]; [s q]
```

$$[\ 0.8988813695351537\ \ 0.9783645344944308\ ] \tag{3}$$

Slightly sublinear convergence is obtained, $q = 0.98$, and the rate of convergence $s \cong 0.90$.

```
∴ x=n; y=d; clf(); semilogy(x,y,"."); grid("on");
```
```
∴ xlabel(L"$n$"); ylabel(L"$d_n$"); title("Convergence␣plot");
```
```
∴ savefig(homedir()*"/courses/MATH661/images/E02Fig03.eps");
```
```
∴
```

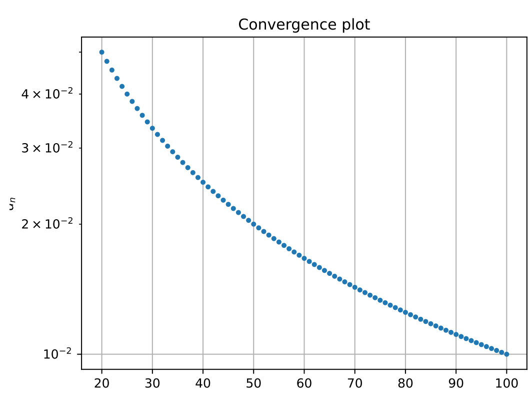

Figure 3. Sub-linear convergence behavior of the sequence $x_n = 1/n$.

Exercise 6. Superimpose on your previous plot the convergence plot of Aitken acceleration applied to $x_n = 1/n$. What order of convergence is achieved?

Solution. Assuming that $x_n \to x$ linearly, Aitken acceleration considers two successive steps

$$x_{n+2} - x \cong r(x_{n+1} - x), x_{n+1} - x \cong r(x_n - x) \Rightarrow x \cong \frac{x_n x_{n+2} - x_{n+1}^2}{x_n - 2x_{n+1} + x_{n+2}}. \tag{4}$$

The above states that from data $\{x_n, x_{n+1}, x_{n+2}\}$, i.e., three successive sequence terms, an estimate of the limit as $n \to \infty$ is obtained. Since this *predicts* the future, the estimate is called an *extrapolation*. Good extrapolations are notoriously hard to obtain, so (4) should not be assumed to be a precise estimate of the limit, but rather the object of further numerical experimentation and mathematical analysis. The approximation (4) can be rewritten as

$$x \cong x_n - \frac{(x_{n+1} - x_n)^2}{x_n - 2x_{n+1} + x_{n+2}} \cong x_{n+2} - \frac{(x_{n+2} - x_{n+1})^2}{x_n - 2x_{n+1} + x_{n+2}}.$$

The difficulties that arise in extrapolation can be highlighted by computation of the condition number of the problem $f: \mathbb{R}^3 \to \mathbb{R}$

$$f(x_n, x_{n+1}, x_{n+2}) = x_n - \frac{(x_{n+1} - x_n)^2}{x_n - 2x_{n+1} + x_{n+2}}.$$

The appearance of a difference of nearly equal quantities in the denominator hints at ill-conditioning, and since f is differentiable the condition number can be defined as the norm of the gradient $\|\nabla f\|$. Consider one of the gradient components

$$\frac{\partial f}{\partial x_{n+1}} = \frac{2(x_{n+1} - x_n)(x_{n+1} - x_{n+2})}{(x_n - 2x_{n+1} + x_{n+2})^2} = \frac{-2(x_{n+1} - x_n)(x_{n+2} - x_{n+1})}{[(x_n - x_{n+1}) + (x_{n+2} - x_{n+1})]^2},$$

with shifted indices

$$\frac{\partial f}{\partial x_n} = \frac{2(x_n - x_{n-1})(x_n - x_{n+1})}{(x_{n-1} - 2x_n + x_{n+1})^2} = \frac{-2(x_n - x_{n-1})(x_{n+1} - x_n)}{[(x_{n-1} - x_n) + (x_{n+1} - x_n)]^2} = \frac{2 b_n c_n}{(c_n - b_n)^2}$$

and, in the spirit of experimentation, evaluate this component for the $x_n = 1/n$ sequence

```
∴ M=20; N=100; n=M:N; M1=M+1; N1=N-1; x=1 ./ n;
```

```
∴ b[M1:N1]=x[M1:N1]-x[M1-1:N1-1]; c[M1:N1]=x[M1+1:N1+1]-x[M1:N1];
```

```
BoundsError([0.05, 0.047619047619047616, 0.045454545454545456,
0.043478260869565216, 0.041666666666666664, 0.04, 0.038461538461538464,
0.037037037037037035, 0.03571428571428571, 0.034482758620689655,
0.03333333333333333, 0.03225806451612903, 0.03125, 0.030303030303030304,
0.029411764705882353, 0.02857142857142857, 0.027777777777777776,
0.02702702702702703, 0.02631578947368421, 0.02564102564102564, 0.025,
0.024390243902439025, 0.023809523809523808, 0.023255813953488372,
0.022727272727272728, 0.022222222222222223, 0.021739130434782608,
0.02127659574468085, 0.020833333333333332, 0.02040816326530612, 0.02,
0.0196078431372549, 0.019230769230769232, 0.018867924528301886,
0.018518518518518517, 0.01818181818181818, 0.017857142857142856,
0.017543859649122806, 0.017241379310344827, 0.01694915254237288,
0.016666666666666666, 0.01639344262295082, 0.016129032258064516,
0.015873015873015872, 0.015625, 0.015384615384615385, 0.015151515151515152,
0.014925373134328358, 0.014705882352941176, 0.014492753623188406,
0.014285714285714285, 0.014084507042253521, 0.013888888888888888,
0.0136986301369863, 0.013513513513513514, 0.013333333333333334,
0.013157894736842105, 0.012987012987012988, 0.01282051282051282,
0.012658227848101266, 0.0125, 0.012345679012345678, 0.012195121951219513,
0.012048192771084338, 0.011904761904761904, 0.011764705882352941,
0.011627906976744186, 0.011494252873563218, 0.011363636363636364,
0.011235955056179775, 0.011111111111111112, 0.01098901098901099,
0.010869565217391304, 0.010752688172043012, 0.010638297872340425,
0.010526315789473684, 0.010416666666666666, 0.010309278350515464,
0.01020408163265306, 0.010101010101010102, 0.01], (21:99,))
```

```
∴ d=a; lgd=log10.(d);
```

```
∴ m=N-M; y=lgd[2:m+1]; A=ones(m,2); A[:,2]=lgd[1:m];
```

```
∴ x=A\y; s=10^x[1]; q=x[2]; [s q]
```

$$[\ 0.8988813695351537 \ \ 0.9783645344944308\ ] \tag{5}$$

```
∴
```

Slightly sublinear convergence is obtained, $q = 0.98$, and the rate of convergence $s \cong 0.89$.

```
∴ x=n; y=d; clf(); semilogy(x,y,"."); grid("on");
```

```
∴ xlabel(L"$n$"); ylabel(L"$d_n$"); title("Convergence␣plot");
```

```
∴ savefig(homedir()*"/courses/MATH661/images/E02Fig03.eps");
```

```
∴
```

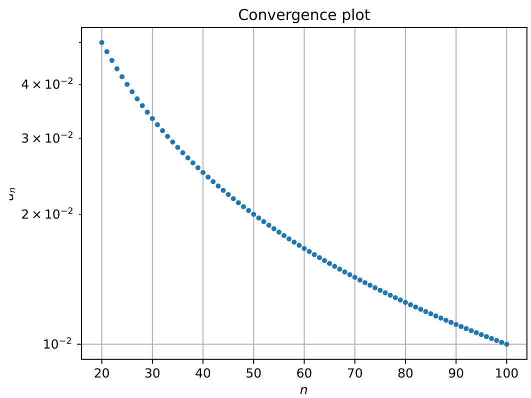

Figure 4. Sub-linear convergence behavior of the sequence $a_n = 1/n$.

Exercise 7. What is the order of convergence of the sequence $x_n = 1/n^2$? Construct a convergence plot.

Exercise 8. Superimpose on your previous plot the convergence plot of Aitken acceleration applied to $x_n = 1/n^2$. What order of convergence is achieved?

3. Approximation techniques

Exercise 9. Find six significant digits of $\sqrt{2}$ through the continued fraction

$$\sqrt{2} = 1 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{2}$$

Exercise 10. Consider the sequence $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}, n \in \mathbb{N} \setminus \{0\}$. Express a_n through function composition

$$a_n = T_n(z), T_n = t_0 \circ t_1 \circ \ldots \circ t_n.$$

Exercise 11. Express the Wallis product through function composition.

Exercise 12. Find the order of convergence of the Ramanujan series.