# LINEAR ALGEBRA TOOLS - EXERCISES

## 1. Norms

**Exercise 1.** Prove the Hölder inequality: for $p, q > 1$, $1/p + 1/q = 1$,

$$\sum_{i=1}^{m} |x_i y_i| \leqslant \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{m} |y_i|^q\right)^{1/q}.$$

**Exercise 2.** Prove the Minkowski inequality: for $p \geqslant 1$,

$$\left(\sum_{i=1}^{m} |x_i + y_i|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{m} |y_i|^p\right)^{1/p}.$$

**Exercise 3.** Prove the parallelogram identity

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2),$$

for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^m$, with $\| \|$ denoting the 2-norm.

**Exercise 4.** Consider $\boldsymbol{A} \in \mathbb{C}^{m \times m}$, $C(\boldsymbol{A}) = \mathbb{C}^m$. Prove that

$$(\boldsymbol{x}^* \boldsymbol{A}^* \boldsymbol{A} \boldsymbol{x})^{1/2}$$

is a norm.

**Exercise 5.** For $\boldsymbol{x} \in \mathbb{R}^m$, prove $\|\boldsymbol{x}\|_\infty \leqslant \|\boldsymbol{x}\|_2$.

**Exercise 6.** For $\boldsymbol{x} \in \mathbb{R}^m$, prove $\|\boldsymbol{x}\|_2 \leqslant \sqrt{m}\, \|\boldsymbol{x}\|_\infty$.

**Exercise 7.** For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, prove $\|\boldsymbol{A}\|_\infty \leqslant \sqrt{n}\, \|\boldsymbol{A}\|_2$.

**Exercise 8.** For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, prove $\|\boldsymbol{A}\|_2 \leqslant \sqrt{m}\, \|\boldsymbol{A}\|_\infty$.

**Exercise 9.** For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{B} \in \mathbb{R}^{p \times q}$, $p \leqslant m$, $q \leqslant n$, $\boldsymbol{B}$ a submatrix of $\boldsymbol{A}$, prove $\|\boldsymbol{B}\|_p \leqslant \|\boldsymbol{A}\|_p$, for any $p$, $1 \leqslant p \leqslant \infty$.

## 2. Projection and orthogonality

**Exercise 10.** Consider $\boldsymbol{u}, \boldsymbol{v} \in V$, $\mathscr{V} = (V, \mathbb{R}, +, \cdot)$ a vector space with norm induced by a scalar product $\|\boldsymbol{u}\|^2 = (\boldsymbol{u}, \boldsymbol{u})$. Prove that $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| \Rightarrow (\boldsymbol{u}+\boldsymbol{v}) \perp (\boldsymbol{u}-\boldsymbol{v})$. Is the converse true?

**Exercise 11.** Let $\mathscr{B} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$ be an orthonormal basis for $\mathscr{V} = (V, \mathbb{R}, +, \cdot)$ with norm induced by a scalar product. For $\boldsymbol{x} = \sum_{i=1}^{m} \xi_i \boldsymbol{u}_i$, prove

$$\sum_{i=1}^{m} |\xi_i|^2 = \|\boldsymbol{x}\|^2.$$

**Exercise 12.** Let $\mathscr{C} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ be an orthonormal set for $\mathscr{V} = (V, \mathbb{R}, +, \cdot)$ with norm induced by a scalar product. For $\boldsymbol{x} = \sum_{i=1}^{m} \xi_i \boldsymbol{u}_i$, prove

$$\sum_{i=1}^{n} |\xi_i|^2 = \|\boldsymbol{x}\|^2.$$

**Exercise 13.** For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(\boldsymbol{A}) = n \leqslant m$, prove the $QR$ factorization $\boldsymbol{Q}\boldsymbol{R} = \boldsymbol{A}$ is unique with $\boldsymbol{Q}$ orthogonal, $\boldsymbol{R}$ upper triangular.

**Exercise 14.** For $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ skew-symmetric prove that $\boldsymbol{U} = (\boldsymbol{I}-\boldsymbol{A})(\boldsymbol{I}-\boldsymbol{A})^{-1} = (\boldsymbol{I}-\boldsymbol{A})^{-1}(\boldsymbol{I}-\boldsymbol{A})$, and $\boldsymbol{U}$ is unitary.

**Exercise 15.** Prove that if $\boldsymbol{P}$ is an orthogonal projector then $\boldsymbol{I} - 2\boldsymbol{P}$ is unitary.