# LINEAR ALGEBRA EXERCISES

**Exercise 1.** Prove that for $\boldsymbol{U}$ unitary $\|\boldsymbol{U}\| = 1$

**Exercise 2.** Compute the singular value decomposition of $\boldsymbol{R}_2(\theta) \in \mathbb{R}^{2\times 2}$, a rotation matrix in the $x_1 x_2$-plane.

**Exercise 3.** Compute the singular value decomposition of $\boldsymbol{R}_{3,1}(\theta) \in \mathbb{R}^{3\times 3}$, the rotation matrix in $\mathbb{R}^3$ around the $x_1$-axis.

**Exercise 4.** Compute the singular value decomposition of $\boldsymbol{R}_{m,j}(\theta) \in \mathbb{R}^{m\times m}$, the rotation matrix in $\mathbb{R}^m$ around the $x_j$-axis.

**Exercise 5.** Establish the geometric significance of the matrix $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}_{m,j}(\theta)$, $\boldsymbol{Q}$ unitary, $\boldsymbol{R}_{m,j}(\theta)$ from above.

**Exercise 6.** Compute the singular value decomposition of $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}_{m,j}(\theta)$, $\boldsymbol{Q}$ unitary, $\boldsymbol{R}_{m,j}(\theta)$ from above.

**Exercise 7.** Prove that the singular values of $\boldsymbol{A}$ hermitian are the absolute value of its eigenvalues.

**Exercise 8.** For $\boldsymbol{A} \in \mathbb{C}^{m\times m}$ invertible with singular value decomposition $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^*$, determine the SVD of $\boldsymbol{A}^{-1}$ in terms of $\boldsymbol{U}, \boldsymbol{\Sigma}, \boldsymbol{V}$.

**Exercise 9.** For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{B} \in \mathbb{R}^{p\times q}$, $p \leqslant m$, $q \leqslant n$, $\boldsymbol{B}$ a submatrix of $\boldsymbol{A}$, prove $\|\boldsymbol{B}\|_p \leqslant \|\boldsymbol{A}\|_p$, for any $p$, $1 \leqslant p \leqslant \infty$.

**Exercise 10.** Consider $\boldsymbol{u}, \boldsymbol{v} \in V$, $\mathscr{V} = (V, \mathbb{R}, +, \cdot)$ a vector space with norm induced by a scalar product $\|\boldsymbol{u}\|^2 = (\boldsymbol{u}, \boldsymbol{u})$. Prove that $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| \Rightarrow (\boldsymbol{u}+\boldsymbol{v}) \perp (\boldsymbol{u}-\boldsymbol{v})$. Is the converse true?

**Exercise 11.** Let $\mathscr{B} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$ be an orthonormal basis for $\mathscr{V} = (V, \mathbb{R}, +, \cdot)$ with norm induced by a scalar product. For $\boldsymbol{x} = \sum_{i=1}^m \xi_i \boldsymbol{u}_i$, prove

$$\sum_{i=1}^{m} |\xi_i|^2 = \|\boldsymbol{x}\|^2.$$

**Exercise 12.** Let $\mathscr{C} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ be an orthonormal set for $\mathscr{V} = (V, \mathbb{R}, +, \cdot)$ with norm induced by a scalar product. For $\boldsymbol{x} = \sum_{i=1}^m \xi_i \boldsymbol{u}_i$, prove

$$\sum_{i=1}^{n} |\xi_i|^2 = \|\boldsymbol{x}\|^2.$$

**Exercise 13.** For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(\boldsymbol{A}) = n \leqslant m$, prove the $QR$ factorization $\boldsymbol{Q}\boldsymbol{R} = \boldsymbol{A}$ is unique with $\boldsymbol{Q}$ orthogonal, $\boldsymbol{R}$ upper triangular.

**Exercise 14.** For $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ skew-symmetric prove that $\boldsymbol{U} = (\boldsymbol{I}-\boldsymbol{A})(\boldsymbol{I}-\boldsymbol{A})^{-1} = (\boldsymbol{I}-\boldsymbol{A})^{-1}(\boldsymbol{I}-\boldsymbol{A})$, and $\boldsymbol{U}$ is unitary.

**Exercise 15.** Prove that if $\boldsymbol{P}$ is an orthogonal projector then $\boldsymbol{I} - 2\boldsymbol{P}$ is unitary.