# EIGENPROBLEM EXERCISES

Notation: $\boldsymbol{A} \in \mathbb{C}^{m\times m}$, $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}, \boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}$ the eigenproblem, $\boldsymbol{A}\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{\Lambda}$ the eigenproblem for $\boldsymbol{A}$ normal ($\boldsymbol{A} = \boldsymbol{A}^*$), $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^*$, the SVD,

$$\boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1,\dots,\lambda_m), \boldsymbol{X} = [\ \boldsymbol{x}_1\ \ \boldsymbol{x}_2\ \ \dots\ \ \boldsymbol{x}_m\ ], \boldsymbol{Q} = [\ \boldsymbol{q}_1\ \ \boldsymbol{q}_2\ \ \dots\ \ \boldsymbol{q}_m\ ],$$

$$p_{\boldsymbol{A}}(\lambda) = \det(\lambda\boldsymbol{I} - \boldsymbol{A}) = (\lambda-\lambda_1)\cdot\ldots\cdot(\lambda-\lambda_m) = \lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_0,$$

$$p_{\boldsymbol{A}}(\boldsymbol{B}) = \boldsymbol{B}^m + a_{m-1}\boldsymbol{B}^{m-1} + \cdots + a_0\boldsymbol{B}^0.$$

**Exercise 1.** Prove $\boldsymbol{\Lambda} = \boldsymbol{0} \Rightarrow \boldsymbol{A} = \boldsymbol{0}$.

**Exercise 2.** Find $\boldsymbol{X}, \boldsymbol{\Lambda}$ for $\boldsymbol{A} = [1]$ (all elements are one).

**Exercise 3.** Let $\lambda_1 \neq \lambda_2$, prove $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ is not an eigenvector of $\boldsymbol{A}$.

**Exercise 4.** For $\boldsymbol{x} \in \mathbb{R}^m, \boldsymbol{A} \in \mathbb{R}^{m\times m}$, is $\boldsymbol{u} = c\boldsymbol{x}$ an eigenvector for $c \in \mathbb{C}$?

**Exercise 5.** Prove that for $\boldsymbol{A} = \boldsymbol{A}^*$, $|\lambda|$ is a singular value.

**Exercise 6.** Consider the mapping $S : \mathbb{C}^m \to \mathbb{C}^m$

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{m-1} \\ v_m \end{bmatrix}, S(\boldsymbol{v}) = \begin{bmatrix} v_2 \\ v_3 \\ \vdots \\ v_m \\ v_1 \end{bmatrix}.$$

1. Prove $S$ is a linear map. Find its matrix representation, $\boldsymbol{A}$.
2. Prove $\boldsymbol{A}$ is orthogonal.
3. Prove that the eigenvectors of $\boldsymbol{A}$ are

$$\boldsymbol{x}_k = \begin{bmatrix} \omega^{k\cdot 0} \\ \omega^{k\cdot 1} \\ \vdots \\ \omega^{k\cdot(m-1)} \end{bmatrix}, \omega = \exp\left(\frac{2\pi i}{m}\right).$$

4. Find the eigenvalues $\boldsymbol{A}\boldsymbol{x}_k = \lambda_k\boldsymbol{x}_k$

**Exercise 7.** Let $\rho(\boldsymbol{A}) = \max_k\{|\lambda_k|\}$ (the spectral radius of $\boldsymbol{A}$). Prove

$$\lim_{n\to\infty} \|\boldsymbol{A}^n\|_2 = 0 \Leftrightarrow \rho(A) < 1,$$

**Exercise 8.** Why do similar matrices have the same eigenvalues? Give both a proof and an intuitive explanation.

**Exercise 9.** Consider $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}$, $\boldsymbol{B}\boldsymbol{Y} = \boldsymbol{Y}\boldsymbol{\Gamma}$, $\boldsymbol{A} \in \mathbb{C}^{m\times m}$, $\boldsymbol{B} \in \mathbb{C}^{n\times n}$. Find the eigenvalues and eigenvectors of

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{bmatrix}.$$

**Exercise 10.** Prove that eigenvalues of $\boldsymbol{A}$ skew-symmetric ($\boldsymbol{A} = -\boldsymbol{A}^*$) are purely imaginary ($\operatorname{Re}\lambda = 0$).

**Exercise 11.** Let $\boldsymbol{D}$ denote a diagonal matrix. Prove that $p_{\boldsymbol{D}}(\boldsymbol{D}) = \boldsymbol{0}$.

**Exercise 12.** Prove that for $\boldsymbol{A}$ non-defective $p_{\boldsymbol{A}}(\boldsymbol{A}) = \boldsymbol{0}$.

**Exercise 13.** Prove the Cayley-Hamilton theorem (generalization of Ex.11, Ex. 12), $p_{\boldsymbol{A}}(\boldsymbol{A}) = \boldsymbol{0}$.

**Exercise 14.** For $\boldsymbol{A}$ normal, prove $\boldsymbol{A} - \lambda\boldsymbol{I}$ is normal.

**Exercise 15.** For $\boldsymbol{A}$ normal, prove that $\boldsymbol{A}, \boldsymbol{A}^*$ have the same eigenvectors.

**Exercise 16.** Prove $\operatorname{tr}(\boldsymbol{A}) = \sum_{j=1}^m a_{jj}$.

**Exercise 17.** Find all $\lambda \in \mathbb{C}$ such that $\boldsymbol{I} - \lambda\boldsymbol{u}\boldsymbol{u}^*$ is unitary for some $\boldsymbol{u} \neq \boldsymbol{0}$.

**Exercise 18.** For $\boldsymbol{x} \in \mathbb{R}^m, \boldsymbol{A} \in \mathbb{R}^{m\times m}$ is $\boldsymbol{u} = c\boldsymbol{x}$ an eigenvector for $c \in \mathbb{C}$?

**Exercise 19.** Find the characteristic polynomial of

$$\boldsymbol{A} = \begin{bmatrix} a_1 & a_2 & \dots & a_{m-1} & a_m \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ & & \ddots & & \\ & & & 1 & 0 \end{bmatrix}$$

**Exercise 20.** Let $\boldsymbol{A}$ be tridiagonal with non-zero subdiagonal entries. Prove that eigenvalues of $\boldsymbol{A}$ are distinct.