

## APPROXIMATION EXERCISES

Ex1-3: Find the polynomial of least degree that interpolates the following data  $\mathcal{D} = \{(x_i, y_i, y_i', y_i''), i = 0, 1, \dots, n\}$ .

**Exercise 1.** Find the polynomial of least degree that interpolates the data  $\mathcal{D} = \{(x_i, y_i), i = 0, 1, \dots, n\} = \{(3, 10), (7, 146), (1, 2), (2, 1)\}$ .

**Exercise 2.** Find the polynomial of least degree that interpolates the data  $\mathcal{D} = \{(x_i, y_i, y_i'), i = 0, 1, \dots, n\} = \mathcal{D} = \{(3, 10, 14), (1, 2, -6)\}$ .

**Exercise 3.** Find the polynomial of least degree that interpolates the data  $\mathcal{D} = \{(x_i, y_i, y_i', y_i''), i = 0, 1, \dots, n\} = \mathcal{D} = \{(1, 2, -6, 10)\}$ .

**Exercise 4.** Consider  $\mathcal{F} = \{f \mid f: [a, b] \rightarrow \mathbb{R}\}$ ,  $\mathcal{P}_n = \{p \mid p(t) = a_0 + a_1 t + \dots + a_n t^n, x \in \mathbb{R}^{n+1}, x_i \in [a, b], i \neq j \Rightarrow x_i \neq x_j\}$ . Define the operators  $S: \mathcal{F} \rightarrow \mathbb{R}^{(n+1) \times 2}$ ,  $I: \mathbb{R}^{(n+1) \times 2} \rightarrow \mathcal{P}_n \subset \mathcal{F}$ ,

$$f \in \mathcal{F}, \mathbf{y} = f(\mathbf{x}) \in \mathbb{R}^{n+1}, S(f) = [\mathbf{x} \ \mathbf{y}],$$

$$I([\mathbf{x} \ \mathbf{y}]) = p_n(t), p_n(\mathbf{x}) = \mathbf{y} = f(\mathbf{x}).$$

For each of the mappings,  $S, I, P = I \circ S$ , determine if the mapping is linear, and if so write its matrix representation. Also determine if the mappings are surjective, bijective, one-to-one.

**Exercise 5.** Recall the eigenvalue problem for finite-dimensional linear operators,  $A: \mathbb{R}^m \rightarrow \mathbb{R}^m, A\mathbf{X} = \mathbf{X}\Lambda, \mathbf{X} \in \mathbb{R}^{m \times m}, \Lambda \in \mathbb{C}^{m \times m}, \Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$ . The generalization of the finite-dimensional problem to linear operators on vector space  $\mathcal{U}, L: \mathcal{U} \rightarrow \mathcal{U}$  is immediate,  $L\mathbf{u} = \lambda\mathbf{u}, \mathbf{u} \in \mathcal{U}, \lambda \in \mathbb{C}$ . What is the appropriate generalization of  $C(\mathbf{X})$  for the operator  $P: \mathcal{F} \rightarrow \mathcal{F}$ ?

**Exercise 6.** Provide a relative error bound for the polynomial interpolation of  $f: [-1, 1] \rightarrow \mathbb{R}, f(t) = \cosh(t)$  by a polynomial  $p_{22}(t)$  of degree  $n = 22$ .

**Exercise 7.** Prove that the above bound  $\varepsilon \leq 5 \times 10^{-16}$ , irrespective of the choice of distinct interpolation nodes.

**Exercise 8.** In the limit  $x_1 \rightarrow x_0$  the divided difference

$$f[x_0, x_1] = [y_0, y_1] = \frac{y_1 - y_0}{x_1 - x_0}, y_i = f(x_i), i = 0, 1,$$

has limit  $f[x_0, x_1] \rightarrow f'(x_0)$ . Write and establish the validity of the finite difference form of the product rule  $(fg)' = f'g + fg'$ .

**Exercise 9.** Repeat the above for second order finite differences and  $(fg)'' = f''g + 2f'g' + fg''$ .

**Exercise 10.** A natural cubic spline has zero curvature at the end points. Prove that of all cubic spline interpolations of data  $\mathcal{D} = \{(x_i, y_i = f(x_i)), i = 0, 1, \dots, n\}$ , the natural spline  $S(t)$  curvature two-norm is bounded by the function curvature two-norm

$$\int_{x_0}^{x_n} [S''(t)]^2 dt \leq \int_{x_0}^{x_n} [f''(t)]^2 dt.$$

**Exercise 11.** Recall the behavior of global polynomial interpolation for the Runge function, and interpret the significance of the above inequality for spline interpolation. In particular study the bound for the curvature of the global polynomial interpolant

$$\int_{x_0}^{x_n} [S''(t)]^2 dt.$$

**Exercise 12.** Prove that best inf-norm approximant of  $f: [-1, 1] \rightarrow \mathbb{R}, f(t) = \cosh(t)$  by a quadratic polynomial has form  $p_2(t) = a + bt^2$ , with  $b = \cosh 1 - 1$ . Compute  $a$ .

**Exercise 13.** Compare the error bound for  $p_2(t)$  from Ex 12 with that for  $p_{22}(t)$  from Ex. 6.

**Exercise 14.** Prove that

$$\int_{-1}^1 f(t) dt \cong \frac{1}{90} \left[ 7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1) \right]$$

is exact for  $f \in \mathcal{P}_4$ , (polynomials of degree up to 4).

**Exercise 15.** Determine the quadrature formula

$$\int_0^1 f(t) dt \cong c_0 f(0) + c_1 f(1)$$

that is exact for all functions of form  $f(t) = ae^t + b \cos(\pi t/2)$ .

**Exercise 16.** Determine the quadrature formula

$$\int_0^{2\pi} f(t) dt \cong c_0 f(0) + c_1 f(1)$$

that is exact for all functions of form  $f(t) = a + b \cos(t)$ .

**Exercise 17.** Verify that the above formula is exact for

$$f(t) = \sum_{k=0}^n [a_k \cos(2k+1)t + b_k \sin kt]$$

**Exercise 18.** Can the quadrature

$$\int_0^1 f(t) dt \cong \alpha[f(x_0) + f(x_1)]$$

be exact for all quadratic polynomials?

**Exercise 19.** Let the quadrature

$$\int_{-1}^1 f(t) dt \cong \sum_{i=0}^n c_i f(x_i)$$

be exact for all polynomials of degree  $n = 2k$ ,  $k \in \mathbb{N}$ . Show that if the quadrature nodes  $x_i$  are symmetric around the origin the formula is also exact for polynomials of degree  $n + 1$ .

**Exercise 20.** Determine the minimum number of subintervals needed to approximate

$$\int_1^2 (t + e^{-t^2}) dt$$

to absolute accuracy  $e < 0.5 \times 10^{-7}$  using the composite trapezoid rule.

**Exercise 21.** Repeat above for the composite Simpson rule.

**Exercise 22.** Consider the equation  $f(x) = 0$ ,  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f \in C^{(2)}(\mathbb{R})$ . Generate an approximating sequence  $\{x_n\}_{n \in \mathbb{N}}$ ,  $x_n \rightarrow r$  as  $n \rightarrow \infty$ ,  $f(r) = 0$ , based upon constructing the linear interpolant of data  $\mathcal{D} = \{(x_{n-2}, f_{n-2}), (x_{n-1}, f_{n-1})\}$  to obtain  $x_n$  ( $f_k = f(x_k)$ ).

**Exercise 23.** Consider the equation  $f(x) = 0$ ,  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f \in C^{(2)}(\mathbb{R})$ . Generate an approximating sequence  $\{x_n\}_{n \in \mathbb{N}}$ ,  $x_n \rightarrow r$  as  $n \rightarrow \infty$ ,  $f(r) = 0$ , based upon constructing the linear interpolant of data  $\mathcal{D} = \{(x_{n-1}, f_{n-1}, f'_{n-1})\}$  to obtain  $x_n$  ( $f_k = f(x_k)$ ,  $f'_k = f'(x_k)$ ).

**Exercise 24.** Consider the equation  $f(x) = 0$ ,  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f \in C^{(2)}(\mathbb{R})$ . Generate an approximating sequence  $\{x_n\}_{n \in \mathbb{N}}$ ,  $x_n \rightarrow r$  as  $n \rightarrow \infty$ ,  $f(r) = 0$ , based upon constructing the linear interpolant of data  $\mathcal{D} = \{(x_{n-3}, f_{n-3}), (x_{n-2}, f_{n-2}), (x_{n-1}, f_{n-1})\}$  to obtain  $x_n$  ( $f_k = f(x_k)$ ).

**Exercise 25.** Consider the equation  $f(x) = 0$ ,  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f \in C^{(2)}(\mathbb{R})$ . Generate an approximating sequence  $\{x_n\}_{n \in \mathbb{N}}$ ,  $x_n \rightarrow r$  as  $n \rightarrow \infty$ ,  $f(r) = 0$ , based upon constructing the linear interpolant of data  $\mathcal{D} = \{(x_{n-1}, f_{n-1}, f'_{n-1}, f''_{n-1})\}$  to obtain  $x_n$  ( $f_k = f(x_k)$ ,  $f'_k = f'(x_k)$ ,  $f''_k = f''(x_k)$ ).