# MATH661 EC - Least squares, minimax

**Posted**: 10/25/23
**Due**: 11/08/23, 11:59PM

*Time constraints and clasas cancellation at time of H01 did not allow a computational assignment on least squares and minimax approximation. These are important topics to explore numerically so this assignment is offered as extra credit, with 4 course points for the first three questions. These correspond to a normal homework assignment effort. An additional 3 course points are awarded to correct solution of the fourth question on the Remez algorithm. For Track 1 a solution is to be sought by computer-aided calculation, and for Track 2 a general implementation is required. The effort requried for this single question is comparable to a full homework assignment.*

## 1 Track 1

1. Apply the Gram-Schmidt process to $\{1, t, t^2\}$ to obtain the first three Legendre polynomials $P_0, P_1, P_2$, orthonormal with respect to the scalar product

$$(f, g) = \int_{-1}^{1} f(t)\, g(t)\, \mathrm{d}t.$$

   Show intermediate steps. Check hand calculations with symbolic integration packages.

2. Find the best approximant of $f(t) = 1 + \cos(\pi t) + \sin(\pi t)$ in the Hilbert space $\mathcal{F} = (C^\infty(\mathbb{R}), \mathbb{R}, +, \cdot)$ with scalar product and norm

$$(f, g) = \int_{-1}^{1} f(t)\, g(t)\, \mathrm{d}t, \|f\| = (f, f)^{1/2}.$$

3. Find the best inf-norm approximant by $g(t) = a + bt$ of $f(t) = 1 + \cos(\pi t) + \sin(\pi t)$, $f\colon [0, 1]$.

   a) By the equioscillation theorem, the solution is a line that intersects the graph of $f$ twice such that the error $e(t) = f(t) - g(t)$ has three extrema of equal absolute value at $0, x, 1$, with $e$ stationary at $t = x$. Write the four equations that arise

   b) Solve the above problem, and plot $f, g$.

4. Extra credit (3 points). Apply the Remez algorithm to Q3

## 2 Track 2

1. Find and plot the first six members of the orthonormal basis $\mathcal{L}$ obtained from applying the Gram-Schmidt algorithm to $\{1, t, t^2, \ldots\}$ with the scalar product

$$(f, g) = -\int_{0}^{1} f(t)\, g(t) \log(1 - t)\, \mathrm{d}t.$$

2. Find and plot the best approximants of

$$f(t) = \tan\left(\frac{\pi t}{2}\right),$$

in span($\mathcal{L}_n$) for $\mathcal{L}_n = \{l_0(t), l_1(t), ..., l_n(t)\}$ and scalar product from problem 1, for $n = 1, 2, 3, 4, 5, 6$.

a) First, carry out analytical computations. Symbolic computation software (Maple, Mathematica, Maxima) is allowed. Maxima is available in the SciComp@UNC virtual machine.

b) For each $n$, form the matrix $\boldsymbol{L} = [\ l_0(\boldsymbol{t}) \ \ l_1(\boldsymbol{t}) \ \ ... \ \ \boldsymbol{l}_n(\boldsymbol{t})\ ] \in \mathbb{R}^{m \times n}$ at sample points $\boldsymbol{t}$ within [0,1] and $m \gg n$. Solve the least squares problem

$$\min_{\boldsymbol{c}} \|\boldsymbol{L}\boldsymbol{c} - f(\boldsymbol{t})\|.$$

in the Euclidean two-norm. Plot the numerically determined approximants

$$p(t) = \mathcal{L}_n(t)\, \boldsymbol{c}$$

together with the analytically determined approximants, and compare.

3. Apply the Remez algorithm to find the best inf-norm approximant by a quadratic of $f$: $[0, 1] \to \mathbb{R}$, $f(t) = \sin(\pi t)$. Establish your own convergence criteria.

4. Extra credit (3points). Implement the Remez algorithm. Use whatever root-finding routine is available within the programming environment (e.g., Roots for Julia) to solve the extrema conditions $f'(x) - p'(x)$ needed in the Remez algorithm.