# MATH661/2 Revised Lesson Plan

## 1 MATH661

| Week | Topic |
|---|---|
| 1 | Floating point arithmetic. Approximating sequences. Order of convergence. Summation order and floating point non-associativity. Finite difference approximation of derivative and catastrophic loss of precision. Condition number. |
| 2 | Linear combinations. Vector spaces and subspaces. Bases. Dimension. Orthogonal matrices. Matrix subspaces. Fundamental theorem of linear algebra. Rank-nullity. |
| 3 | Vector and matrix norms. Singular value decomposition theorem & proof. Link to Karhunen-Loève. Rank-1 expansions. Operator approximation. |
| 4 | Linear statement of principal applied mathematics problems: coordinate changes (linear systems), reduced-order models (least squares), operator invariants (eigenproblems). Solution by the SVD. Pseudo-inverse. Midterm 1. |
| 5 | Additional operator representations: $QR$, $LU$, $LL^*$, $QTQ^*$. Computational complexity. |
| 6 | Projection: Householder, Givens. Hessenberg form. $QR$ least-squares solution. Stability |
| 7 | Interpolation in the monomial basis. Newton form (matrix triangularization). Lagrange form, including barycentric. Continuum limits. Taylor series. Residuals. Finite difference calculus. |
| 8 | Interpolation in $B$-spline basis. Bezier curves and surfaces. Computational geometry. Simplicia. Midterm 2. |
| 9 | Approximation in monomial basis. Continuum norms. Stationarity conditions & extrema. $L_1, L_2, L_\infty$ approximants (momentum, energy, constraints of physical systems). |
| 10 | Interpolation in orthogonal bases: trigonometric, Legendre, Laguerre, Hermite, Bessel, spherical harmonics. Fast decompositions. |
| 11 | Linear operator approximation 1: quadrature ($\int \mathrm{d}x^n$). Newton-Cotes. Moments. Gauss. Convergence. Stability. |
| 12 | Linear operator approximation 2: differentiation ($\mathrm{d}/\mathrm{d}x^n, \nabla, \nabla^2$), linear ODE ($\sum_k a_k \mathrm{d}/\mathrm{d}x^k$). Convergence. Stability. |
| 13 | Non-linear operator approximation 1: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 0$. 0-degree approximant (bisection), 1-degree approximants (secant, Newton). Convergence, fixed points. |
| 14 | Non-linear operator approximation 2: Non-linear operator composition $f = l_n \circ \ldots \circ l_1$, Quasi-linear approximants $\boldsymbol{A} = \boldsymbol{\sigma} \circ \boldsymbol{L}_n \circ \ldots \boldsymbol{\sigma} \circ \boldsymbol{L}_1$. Artificial neural networks. |
| 15 | Non-linear operator approximation 3: $f: \mathbb{R}^n \to \mathbb{R}$, $f(x) = 0$. 0,1,2-degree approximants. Convexity, steepest descent, sampled descent directions (stochastic gradient). Recurrent, generative, adversarial neural networks. |

## 2 MATH 662

| Week | Topic |
|---|---|
| 1 | Mixed-operator approximation 1: Sum of linear operators, linear integro-differential equations |
| 2 | Mixed-operator approximation 2: Sum of linear with non-linear operators, general ODEs ($\sum_k a_k \mathrm{d}/\mathrm{d}x^k + f$, $y' = f(y)$). Linear multistep methods. Stability. Convergence |
| 3 | Mixed-operator approximation 3: Runge-Kutta methods |
| 4 | Mixed-operator approximation 4: Integro-differential equations. Fredholm alternative. |
| 5 | Mixed-operator approximation 5: Fractional calculus approximation. Midterm 1 |
| 6 | Operator invariants. Eigenproblems. Spectral expansion. Characteristic polynomial. Cayley-Hamilton. |
| 7 | Iterative coordinate transforms: Jacobi, Gauss-Seidel, SOR |
| 8 | Iterative reduced order methods: Krylov spaces, Arnoldi & Lanczos factorizations. Conjugate gradient, GMRES. |
| 9 | Iterative operator invariant methods: $QR$, Lanczos, Arnoldi. SVD computation |
| 10 | Random variables, probability distributions, sampling, random number generators |
| 11 | Operator sampling: random choice (Glimm's method, Monte Carlo integration) |
| 12 | Stochastic calculus: stochastic processes, Ito, Stratonovich |
| 13 | SDE: Euler-Maruyama, Milstein, Runge-Kutta |
| 14 | Discrete representations: graphs, graph Laplacian |
| 15 | Differentiation & integration on graphs, graph diffusion equation. |