MATH661 HW04 - Midterm review

Posted: 09/20/23
Due: 09/27/23, 11:59PM

At this point in the course homework has addressed:

- ***HW0.*** *Tools needed for scientific computation (number representation, number approximation techniques, basic coding constructs, an environment for method documentation and reproducible computational experiments).*

- ***HW2.*** *Discretization of continuous functions leads to finite-dimensional vectors that can often be approximated by linear combination of just a few of the basis vectors required for the entire space.*

- ***HW3.*** *Large data sets, readily acquired from observations, can guide selection of vectors within a basis to obtain data compression or efficient data representation through linear combination.*

Homework 4 reinforces analytical skills within the mathematical framework of finite-dimensional vector spaces used for the above topics. Such technical proficiency is just as important as efficient coding. The midterm examination verifies proficiency in such analytical skills

Note: The exercises below contain well-known results, but should be attempted individually and independently, without recourse to references. Simply looking up a proof and transcribing it will not aid understanding nor ensure good results on the midterm examination. If you do not obtain an exercise proof within 10 minutes reread the relevant theoretical material from the lecture notes and then try again for another 15 minutes.

1 Tracks 1 and 2

1. Prove the parallelogram identity

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2+\|\boldsymbol{x}-\boldsymbol{y}\|^2=2(\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2),$$

   for $\boldsymbol{x},\boldsymbol{y}\in\mathbb{C}^m$, with $\| \, \|$ denoting the 2-norm.

2. Consider $\boldsymbol{u}, \boldsymbol{v}\in V$, $\mathcal{V}=(V,\mathbb{R},+,\cdot)$ a vector space with norm induced by a scalar product $\|\boldsymbol{u}\|^2=(\boldsymbol{u},\boldsymbol{u})$. Prove that $\|\boldsymbol{u}\|=\|\boldsymbol{v}\| \Rightarrow (\boldsymbol{u}+\boldsymbol{v})\perp(\boldsymbol{u}-\boldsymbol{v})$. Is the converse true?

3. Consider $\boldsymbol{A}\in\mathbb{R}^{m\times m}$, $C(\boldsymbol{A})=\mathbb{R}^m$. Prove that

$$(\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x})^{1/2}$$

is a norm. (Track 2: generalize above to $\mathbb{C}^m$)

4. Construct the matrix $\boldsymbol{A}$ that represents the mapping $\boldsymbol{f}$: $\mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\boldsymbol{f}$ reflects a vector across the x_1x_2 plane. Construct the matrix $\boldsymbol{B}$ that represents the mapping $\boldsymbol{g}$: $\mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\boldsymbol{g}$ reflects a vector across the x_2x_3 plane. Determine the mapping represented by $\boldsymbol{C} = \boldsymbol{B}\boldsymbol{A}$.

5. Prove that the inverse of a rank-1 perturbation of $\boldsymbol{I}$ is itself a rank-1 perturbation of $\boldsymbol{I}$, namely

$$(\boldsymbol{I} + \boldsymbol{u}\boldsymbol{v}^*)^{-1} = \boldsymbol{I} + \theta \boldsymbol{u}\boldsymbol{v}^*.$$

Determine the scalar θ.

6. Determine the rank of $\boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{u}\boldsymbol{v}^*$.

7. Write the inverse $(\boldsymbol{I} + \boldsymbol{A}^{-1}\boldsymbol{u}\boldsymbol{v}^*)^{-1}$ as a rank-1 perturbation of $\boldsymbol{I}$.

8. Consider $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{u}\boldsymbol{v}^* = \boldsymbol{A}(\boldsymbol{I} + \boldsymbol{A}^{-1}\boldsymbol{u}\boldsymbol{v}^*)$. Write $\boldsymbol{C}^{-1}$ as a rank-1 perturbation of $\boldsymbol{A}^{-1}$.

2 Track 2

1. For $\boldsymbol{x} \in \mathbb{R}^m$, prove $\|\boldsymbol{x}\|_\infty \leqslant \|\boldsymbol{x}\|_2$.

2. For $\boldsymbol{x} \in \mathbb{R}^m$, prove $\|\boldsymbol{x}\|_2 \leqslant \sqrt{m}\, \|\boldsymbol{x}\|_\infty$.

3. For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, prove $\|\boldsymbol{A}\|_\infty \leqslant \sqrt{n}\, \|\boldsymbol{A}\|_2$.

4. For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, prove $\|\boldsymbol{A}\|_2 \leqslant \sqrt{m}\, \|\boldsymbol{A}\|_\infty$.

5. Prove the Minkowski inequality: for $p \geqslant 1$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$, $\|\boldsymbol{x} + \boldsymbol{y}\|_p \leqslant \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p$.

6. Construct the matrix $\boldsymbol{D}$ that represents the mapping $\boldsymbol{f}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\boldsymbol{f}$ rotates a vector around the x_3 axis by angle θ. Construct the matrix $\boldsymbol{E}$ that represents the mapping $\boldsymbol{f}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\boldsymbol{f}$ rotates a vector around the x_2 axis by angle φ.

7. What do $\boldsymbol{D}\boldsymbol{E}$ and $\boldsymbol{E}\boldsymbol{D}$ represent?

8. Is $\boldsymbol{D}\boldsymbol{E} = \boldsymbol{E}\boldsymbol{D}$ true? Explain.