MATH661 HW05 - Linear algebra analytical practice

Posted: 10/04/23
Due: 10/11/23, 11:59PM

While working on computational aspects in P01, homework will concentrate on analytical properties.

1 Track 1

1. Let λ, μ be distinct eigenvalues of $\boldsymbol{A}$ symmetric, i.e., $\lambda \neq \mu$, $\boldsymbol{A} = \boldsymbol{A}^T$, $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$, $\boldsymbol{A}\boldsymbol{y} = \mu \boldsymbol{y}$. Show that $\boldsymbol{x}, \boldsymbol{y}$ are orthogonal.

2. Consider

$$\boldsymbol{A} = \begin{bmatrix} -i & -i & 0 \\ -i & i & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   a) Is $\boldsymbol{A}$ normal?

   b) Is $\boldsymbol{A}$ self-adjoint?

   c) Is $\boldsymbol{A}$ unitary?

   d) Find the eigenvalues and eigenvectors of $\boldsymbol{A}$.

3. Find the eigenvalues and eigenvectors of the matrix $\boldsymbol{R} \in \mathbb{R}^{3\times 3}$ expressing rotation around the $z-$axis (unit vector $\boldsymbol{e}_3 = [\ 0\ \ 0\ \ 1\ ]^T$).

4. Find the eigenvalues and eigenvectors of the matrix $\boldsymbol{R} \in \mathbb{R}^{3\times 3}$ expressing rotation around the axis with unit vector $l = \frac{1}{\sqrt{3}}[\ 1\ \ 1\ \ 1\ ]^T$.

5. Compute $\sin(\boldsymbol{A}t)$ for

$$\boldsymbol{A} = \begin{bmatrix} 3 & -9 \\ 2 & -6 \end{bmatrix}.$$

6. Compute $\cos(\boldsymbol{A}t)$ for

$$\boldsymbol{A} = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}.$$

7. Compute the SVD of

$$\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

   by finding the eigenvalues and eigenvectors of $\boldsymbol{A}\boldsymbol{A}^T$, $\boldsymbol{A}^T\boldsymbol{A}$.

8. Find the eigenvalues and eigenvectors of $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ with elements $a_{ij} = 1$ for all $1 \leqslant i, j \leqslant m$. Hint: start with $m = 1, 2, 3$ and generalize.

2 Track 2

1. Prove that $\boldsymbol{A} \in \mathbb{C}^{m\times m}$ is normal if and only if it has m orthonormal eigenvectors.

2. Prove that $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ symmetric has a repeated eigenvalue if and only if it commutes with a non-zero skew-symmetric matrix $\boldsymbol{B}$.

3. Prove that every positive definite matrix $\boldsymbol{K} \in \mathbb{R}^{m \times m}$ has a unique square root $\boldsymbol{B}$, $\boldsymbol{B}$ positive definite and $\boldsymbol{B}^2 = \boldsymbol{K}$.

4. Find all positive definite orthogonal matrices.

5. Find the eigenvalues and eigenvectors of a Householder reflection matrix.

6. Find the eigenvalues and eigenvectors of a Givens rotation matrix.

7. Prove or state a counterexample: If all eigenvalues of $\boldsymbol{A}$ are zero then $\boldsymbol{A} = \boldsymbol{0}$.

8. Prove: A hermitian matrix is unitarily diagonalizable and its eigenvalues are real.