MATH661 HW07 - Least squares, minimax

Posted: 10/18/23
Due: 10/25/23, 11:59PM

These exercises focus on midterm examination preparation.

1 Track 1

1. Find the polynomial of least degree that interpolates the data $\mathcal{D} = \{(x_i, y_i), i = 0, 1, ..., n\} = \{(3, 10), (7, 146), (1, 2), (2, 1)\}$.

2. Find the polynomial of least degree that interpolates the data $\mathcal{D} = \{(x_i, y_i, y_i'), i = 0, 1, ..., n\} = \mathcal{D} = \{(3, 10, 14), (1, 2, -6)\}$.

3. Find the polynomial of least degree that interpolates the data $\mathcal{D} = \{(x_i, y_i, y_i', y_i''), i = 0, 1, ..., n\} = \mathcal{D} = \{(1, 2, -6, 10)\}$.

4. Find a, b, c, d such that

$$S(x) = \begin{cases} 1 - 2x & x \in (-\infty, -3] \\ a + bx + cx^2 + dx^3 & x \in [-3, 4] \\ 157 - 32x & x \in [4, \infty) \end{cases},$$

is a natural spline on the $[-3, 4]$ interval.

5. Apply the Gram-Schmidt algorithm to orthonormalize the function set $\{1, t, t^2\}$ with respect to the scalar product

$$(f, g) = \int_{-1}^{+1} f(t)\, g(t)\, \mathrm{d}t.$$

6. Let $\{\varphi_0(t), \varphi_1(t), \varphi_2(t)\}$ denote the orthonormalized set found above. Find the best 2-norm approximant $g(t) = c_0\varphi_0(t) + c_1\varphi_1(t) + c_2\varphi_2(t)$ of $f(t) = \sin(\pi t/2)$ on the interval $[-1, 1]$.

7. As above, find the best 2-norm approximant of $f(t) = \cos(\pi t/2)$ on the interval $[-1, 1]$.

8. Find the best approximant $g(t) = \lambda t$ of $f(t) = \sin t$ on the interval $[0, \pi/2]$ in the ∞-norm.

2 Track 2

1. In the limit $x_1 \to x_0$ the divided difference

$$f[x_0, x_1] = [y_1, y_0] = \frac{y_1 - y_0}{x_1 - x_0}, y_i = f(x_i), i = 0, 1,$$

has limit $f[x_0, x_1] \to f'(x_0)$. Write and establish the validity of the finite difference form of the product rule $(fg)' = f'g + fg'$.

2. Repeat the above for second order finite differences and $(fg)'' = f''g + 2f'g' + fg''$.

3. A natural cubic spline has zero curvature at the end points. Prove that of all cubic spline interpolations of data $\mathcal{D}=\{(x_i, y_i = f(x_i)), i=0,1,...,n\}$, the natural spline $S(t)$ curvature two-norm is bounded by the function curvature two-norm

$$\int_{x_0}^{x_n} [S''(t)]^2 \, \mathrm{d}t \leqslant \int_{x_0}^{x_n} [f''(t)]^2 \, \mathrm{d}t.$$

4. Find a, b, c, d such that

$$|e(1)| = |e(0)| \Rightarrow |\cosh 1 - a - b| = |1 - a| \Rightarrow$$

$$S(x) = \begin{cases} 1 - 2x & x \in (-\infty, -3] \\ a + bx + cx^2 + dx^3 & x \in [-3, 4] \\ 157 - 32x & x \in [4, \infty) \end{cases},$$

is a natural spline on the $[-3, 4]$ interval.

5. Present an analysis of the conditioning of quadratic spline interpolation.

6. Apply the Gram-Schmidt algorithm to orthonormalize the function set $\{1, t, t^2\}$ with respect to the scalar product

$$(f, g) = \int_{-1}^{+1} \frac{f(t)\, g(t)}{\sqrt{1 - t^2}} \, \mathrm{d}t.$$

7. Find the best approximant $g(t) = a + bt$ of $f(t) = \sin t$ on the interval $[0, \pi/2]$ in the 2-norm and the ∞-norm.

8. Prove that best inf-norm approximant of f: $[-1, 1] \rightarrow \mathbb{R}$, $f(t) = \cosh(t)$ by a quadratic polynomial has form $p_2(t) = a + bt^2$, with $b = \cosh 1 - 1$. Compute a.