MATH661 HW08 - Linear operator approximation

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The basic idea in linear operator approximation is to apply the exact operator to an approximation of the input. These exercises explore and reinforce this concept.

1 Track 1

1. Use Taylor series expansions to verify the approximations

$$f'(t) \cong \frac{1}{12h}[-f(t+2h)+8f(t+h)-8f(t-h)+f(t-2h)],$$

$$f'(t) \cong \frac{1}{2h}[-3f(t)+4f(t+h)-f(t+2h)].$$

   Determine the error term. Construct the polynomial approximant $p_n(t) \cong f(t)$ whose derivative leads to the above formula. Conduct a convergence study as $h \to 0$ for $f(t) \in \{\sin(\pi t/4), e^{10t}, e^{-10t}\}$ at $t_0 = 1$, and compare the observed order of convergence with the theoretical estimate.

2. As above for

$$f''(t) \cong \frac{1}{12h^2}[-f(t+2h)+16f(t+h)-30f(t)+16f(t-h)-f(t-2h)],$$

$$f'''(t) \cong \frac{1}{h^3}[f(t+3h)-3f(t+2h)+3f(t+h)-f(t)].$$

3. Construct a recursive function `RecInt(a,b,err,f,Q)` that has arguments scalars a, b, err and functions f, Q and approximates

$$I(f) = \int_a^b f(t)\,\mathrm{d}t$$

   through repeated application of quadrature rule $Q(f, a, b)$ according to the algorithm

   Algorithm Recursive quadrature

   RecInt(a,b,err,f,Q)
   $c = a + (b-a)/2$
   $Q_{\mathrm{ab}} = Q(f,a,b)$; $Q_{\mathrm{ac}} = Q(f,a,c)$; $Q_{\mathrm{cb}} = Q(f,c,b)$
   $e = |Q_{\mathrm{ac}} + Q_{\mathrm{cb}} - Q_{\mathrm{ab}}| / |Q_{\mathrm{ac}} + Q_{\mathrm{cb}}|$
   if $e < \mathrm{err}$
      return $Q_{\mathrm{ac}} + Q_{\mathrm{cb}}$
   else
      return RecInt(a,c,err,f,Q)+RecInt(c,b,err,f,Q)

   Test the recursive integration procedure with trapezoid, Simpson, and Gauss-Legendre rules of orders 2,3 on the integral

$$\int_{-1}^{1} \cos\left(\frac{1}{t}\right).$$

For each case, present plots of the integrand and the evaluation points used in the recursive quadrature algorithm. Construct convergence plots by executing the algorithm for various error thresholds ε_k and recording the number of evaluation points n_k. Plot $(\log n_k, \log \varepsilon_k)$ and comment on whether the observed order of convergence is that predicted by theoretical quadrature error estimates.

2 Track 2

1. Use the finite difference expressions of the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{h}\left(\Delta - \frac{1}{2}\Delta^2 + \ldots\right) = \frac{1}{h}\left(\nabla + \frac{1}{2}\nabla^2 + \ldots\right) = \frac{1}{h}\left(\delta - \frac{\delta^3}{24} + \cdots\right)$$

to obtain the approximations.

$$f'(t) \cong \frac{1}{12h}[-f(t+2h) + 8f(t+h) - 8f(t-h) + f(t-2h)],$$

$$f'(t) \cong \frac{1}{2h}[-3f(t) + 4f(t+h) - f(t+2h)].$$

Conduct a convergence study as $h \to 0$ for $f(t) \in \{\sin(\pi t/4), e^{10t}, e^{-10t}\}$ at $t_0 = 1$, and compare the observed order of convergence with theoretical estimates. How do the three functions differ, and what effect does this have on derivative approximation?

2. As above for

$$f''(t) \cong \frac{1}{12h^2}[-f(t+2h) + 16f(t+h) - 30f(t) + 16f(t-h) - f(t-2h)],$$

$$f'''(t) \cong \frac{1}{h^3}[f(t+3h) - 3f(t+2h) + 3f(t+h) - f(t)].$$

Use the series products

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{h^2}\left(\Delta - \frac{1}{2}\Delta^2 + \ldots\right)\left(\Delta - \frac{1}{2}\Delta^2 + \ldots\right).$$

3. Romberg integration is a combination of trapezoid quadrature over decreasing subintervals and Aitken extrapolation. Implement Romberg integration and test on

$$\int_0^1 e^t \cos(\pi t)\,\mathrm{d}t.$$

Present a convergence sutdy. What is the observed order of convergence?