

## LU FACTORIZATION

### 1. Gaussian elimination and row echelon reduction

Suppose now that  $Ax = b$  admits a unique solution. How to find it? We are especially interested in constructing a general procedure, that will work no matter what the size of  $A$  might be. This means we seek an *algorithm* that precisely specifies the steps that lead to the solution, and that we can program a computing device to carry out automatically. One such algorithm is *Gaussian elimination*.

Consider the system

$$\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ 2x_1 - x_2 + x_3 = 2 \\ 3x_1 - x_2 - x_3 = 1 \end{cases}$$

The idea is to combine equations such that we have one fewer unknown in each equation. Ask: with what number should the first equation be multiplied in order to eliminate  $x_1$  from sum of equation 1 and equation 2? This number is called a Gaussian multiplier, and is in this case  $-2$ . Repeat the question for eliminating  $x_1$  from third equation, with multiplier  $-3$ .

$$\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ 2x_1 - x_2 + x_3 = 2 \\ 3x_1 - x_2 - x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_1 + 2x_2 - x_3 = 2 \\ -5x_2 + 3x_3 = -2 \\ -7x_2 + 2x_3 = -5 \end{cases}$$

Now, ask: with what number should the second equation be multiplied to eliminate  $x_2$  from sum of second and third equations. The multiplier is in this case  $-7/5$ .

$$\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ -5x_2 + 3x_3 = -2 \\ -7x_2 + 2x_3 = -5 \end{cases} \Rightarrow \begin{cases} x_1 + 2x_2 - x_3 = 2 \\ -5x_2 + 3x_3 = -2 \\ -\frac{11}{5}x_3 = -\frac{11}{5} \end{cases}$$

Starting from the last equation we can now find  $x_3 = 1$ , replace in the second to obtain  $-5x_2 = -5$ , hence  $x_2 = 1$ , and finally replace in the first equation to obtain  $x_1 = 1$ .

The above operations only involve coefficients. A more compact notation is therefore to work with what is known as the "bordered matrix"

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 1 & 2 \\ 3 & -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & -7 & 2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & -\frac{11}{5} & -\frac{11}{5} \end{pmatrix}$$

Once the above *triangular* form has been obtain, the solution is found by back substitution, in which we seek to form the identity matrix in the first 3 columns, and the solution is obtained in the last column.

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & -\frac{11}{5} & -\frac{11}{5} \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

### 2. LU-factorization

- We have introduced Gaussian elimination as a procedure to solve the linear system  $Ax = b$  ("find coordinates of vector  $b$  in terms of column vectors of matrix  $A$ "),  $x, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times m}$
- We now reinterpret Gaussian elimination as a sequence of matrix multiplications applied to  $A$  to obtain a simpler, upper triangular form.

#### 2.1. Example for $m = 3$

Consider the system  $Ax = b$

$$\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ 2x_1 - x_2 + x_3 = 2 \\ 3x_1 - x_2 - x_3 = 1 \end{cases}$$

with

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & -1 & -1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

We ask if there is a matrix  $L_1$  that could be multiplied with  $A$  to produce a result  $L_1A$  with zeros under the main diagonal in the first column. First, gain insight by considering multiplication by the identity matrix, which leaves  $A$  unchanged

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & -1 & -1 \end{pmatrix}$$

In the first stage of Gaussian multiplication, the first line remains unchanged, so we deduce that  $L_1$  should have the same first line as the identity matrix

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -7 & 2 \end{pmatrix}$$

Next, recall the way Gaussian multipliers were determined: find number to multiply first line so that added to second, third lines a zero is obtained. This leads to the form

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ ? & 0 & 1 \end{pmatrix}$$

Finally, identify the missing entries with the Gaussian multipliers to determine

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

Verify by carrying out the matrix multiplication

$$L_1A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -7 & 2 \end{pmatrix}$$

Repeat the above reasoning to come up with a second matrix  $L_2$  that forms a zero under the main diagonal in the second column

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7/5 & 1 \end{pmatrix}$$

$$L_2L_1A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7/5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -7 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & -11/5 \end{pmatrix} = U$$

We have obtained a matrix with zero entries under the main diagonal (an upper triangular matrix) by a sequence of matrix multiplications.

## 2.2. General $m$ case

From the above, we assume that we can form a sequence of multiplier matrices such that the result is an upper triangular matrix  $U$

$$L_{m-1} \dots L_2 L_1 A = U$$

- Recall the basic operation in row echelon reduction: constructing a linear combination of rows to form zeros beneath the main diagonal, e.g.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}}a_{12} & \dots & a_{2m} - \frac{a_{21}}{a_{11}}a_{1m} \\ 0 & a_{32} - \frac{a_{31}}{a_{11}}a_{12} & \dots & a_{3m} - \frac{a_{31}}{a_{11}}a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} - \frac{a_{m1}}{a_{11}}a_{12} & \dots & a_{mm} - \frac{a_{m1}}{a_{11}}a_{1m} \end{pmatrix}$$

- This can be stated as a matrix multiplication operation, with  $l_{i1} = a_{i1} / a_{11}$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -l_{21} & 1 & 0 & \dots & 0 \\ -l_{31} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -l_{m1} & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} - l_{21}a_{12} & \dots & a_{2m} - l_{21}a_{1m} \\ 0 & a_{32} - l_{31}a_{12} & \dots & a_{3m} - l_{31}a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} - l_{m1}a_{12} & \dots & a_{mm} - l_{m1}a_{1m} \end{pmatrix}$$

DEFINITION. *The matrix*

$$L_k = \begin{pmatrix} 1 & \dots & 0 & \dots & 1 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & -l_{k+1,k} & \dots & 0 \\ 0 & \dots & -l_{k+2,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & -l_{m,k} & \dots & 1 \end{pmatrix}$$

with  $l_{i,k} = a_{i,k}^{(k)} / a_{k,k}^{(k)}$ , and  $A^{(k)} = (a_{i,j}^{(k)})$  the matrix obtained after step  $k$  of row echelon reduction (or, equivalently, Gaussian elimination) is called a Gaussian multiplier matrix.

- For  $A \in \mathbb{R}^{m \times m}$  nonsingular, the successive steps in row echelon reduction (or Gaussian elimination) correspond to successive multiplications on the left by Gaussian multiplier matrices

$$L_{m-1}L_{m-2} \dots L_2L_1A = U$$

- The inverse of a Gaussian multiplier is

$$L_k^{-1} = \begin{pmatrix} 1 & \dots & 0 & \dots & 1 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & l_{k+1,k} & \dots & 0 \\ 0 & \dots & l_{k+2,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & l_{m,k} & \dots & 1 \end{pmatrix} = I - (L_k - I)$$

- From  $(L_{m-1}L_{m-2} \dots L_2L_1)A = U$  obtain

$$A = (L_{m-1}L_{m-2} \dots L_2L_1)^{-1}U = L_1^{-1}L_2^{-1} \dots L_{m-1}^{-1}U = LU$$

- Due to the simple form of  $L_k^{-1}$  the matrix  $L$  is easily obtained as

$$L = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ l_{2,1} & 1 & 0 & \dots & 0 & 0 \\ l_{3,1} & l_{3,2} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{m-1,1} & l_{m-1,2} & l_{m-1,3} & \dots & 1 & 0 \\ l_{m,1} & l_{m,2} & l_{m,3} & \dots & l_{m,m-1} & 1 \end{pmatrix}$$

We will show that this indeed possible if  $Ax = b$  admits a unique solution. Furthermore, the product of lower triangular matrices is lower triangular, and the inverse of a lower triangular matrix is lower triangular (same applies for upper triangular matrices). Introduce the notation

$$L^{-1} = L_{m-1} \dots L_2 L_1$$

and obtain

$$L^{-1}A = U$$

or

$$A = LU$$

The above result permits a basic insight into Gaussian elimination: the procedure depends on "factoring" the matrix  $A$  into two "simpler" matrices  $L, U$ . The idea of representing a matrix as a product of simpler matrices is fundamental to linear algebra, and we will come across it repeatedly.

For now, the factorization allows us to devise the following general approach to solving  $Ax = b$

1. Find the factorization  $LU = A$
2. Insert the factorization into  $Ax = b$  to obtain  $(LU)x = L(Ux) = Ly = b$ , where the notation  $y = Ux$  has been introduced. The system

$$Ly = b$$

is easy to solve by forward substitution to find  $y$  for given  $b$

3. Finally find  $x$  by backward substitution solution of

$$Ux = y$$

### Algorithm Gauss elimination without pivoting

```

for  $s = 1$  to  $m - 1$ 
  for  $i = s + 1$  to  $m$ 
     $t = -a_{is} / a_{ss}$ 
    for  $j = s + 1$  to  $m$ 
       $a_{ij} = a_{ij} + t \cdot a_{sj}$ 
     $b_i = b_i + t \cdot b_s$ 

for  $s = m$  downto  $1$ 
   $x_s = b_s / a_{ss}$ 
  for  $i = 1$  to  $s - 1$ 
     $b_i = b_i - a_{is} \cdot x_s$ 

```

return  $x$

### Algorithm Gauss elimination with partial pivoting

```

 $p = 1:m$  (initialize row permutation vector)
for  $s = 1$  to  $m - 1$ 
   $piv = \text{abs}(a_{p(s),s})$ 
  for  $i = s + 1$  to  $m$ 
     $mag = \text{abs}(a_{p(i),s})$ 
    if  $mag > piv$  then
       $piv = mag; k = p(s); p(s) = p(i); p(i) = k$ 
  if  $piv < \epsilon$  then break("Singular matrix")
   $t = -a_{p(i)s} / a_{p(s)s}$ 

```

```

for  $j = s + 1$  to  $m$ 
     $a_{p(i)j} = a_{p(i)j} + t \cdot a_{p(s)j}$ 
 $b_{p(i)} = b_{p(i)} + t \cdot b_{p(s)}$ 

for  $s = m$  downto 1
     $x_s = b_{p(s)} / a_{p(s)s}$ 
    for  $i = 1$  to  $s - 1$ 
         $b_{p(i)} = b_{p(i)} - a_{p(i)s} \cdot x_s$ 

return  $x$ 

```

Given  $A \in \mathbb{R}^{m \times n}$

Singular value decomposition

Transformation of coordinates

$$U \Sigma V^T = A$$

$$(U \Sigma V^T)x = b \Rightarrow Uy = b \Rightarrow y = U^T b$$

$$\Sigma z = y \Rightarrow z = \Sigma^+ y$$

$$V^T x = z \Rightarrow x = Vz$$

Gram-Schmidt

$$Ax = b$$

$$QR = A$$

$$(QR)x = b \Rightarrow Qy = b, y = Q^T b$$

$$Rx = y \text{ (back sub to find } x)$$

Lower-upper

$$LU = A$$

$$(LU)x = b \Rightarrow Ly = b \text{ (forward sub to find } y)$$

$$Ux = y \text{ (back sub to find } x)$$

### 3. Inverse matrix

By analogy to the simple scalar equation  $ax = b$  with solution  $x = a^{-1}b$  when  $a \neq 0$ , we are interested in writing the solution to a linear system  $Ax = b$  as  $x = A^{-1}b$  for  $A \in \mathbb{R}^{m \times m}$ ,  $x \in \mathbb{R}^m$ . Recall that solving  $Ax = b = Ib$  corresponds to expressing the vector  $b$  as a linear combination of the columns of  $A$ . This can only be done if the columns of  $A$  form a basis for  $\mathbb{R}^m$ , in which case we say that  $A$  is *non-singular*.

DEFINITION 1. For matrix  $A \in \mathbb{R}^{m \times m}$  non-singular the inverse matrix is denoted by  $A^{-1}$  and satisfies the properties

$$AA^{-1} = A^{-1}A = I$$

#### 3.1. Gauss-Jordan algorithm

Computation of the inverse  $A^{-1}$  can be carried out by repeated use of Gauss elimination. Denote the inverse by  $B = A^{-1}$  for a moment and consider the inverse matrix property  $AB = I$ . Introducing the column notation for  $B, I$  leads to

$$A(B_1 \dots B_m) = (e_1 \dots e_m)$$

and identification of each column in the equality states

$$AB_k = e_k, k = 1, 2, \dots, m$$

with  $e_k$  the column unit vector with zero components everywhere except for a 1 in row  $k$ . To find the inverse we need to simultaneously solve the  $m$  linear systems given above.

*Gauss-Jordan algorithm example.* Consider

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 2 & -1 & -2 \end{pmatrix}$$

To find the inverse we solve the systems  $AB_1 = e_1, AB_2 = e_2, AB_3 = e_3$ . This can be done simultaneously by working with the matrix  $A$  bordered by  $I$

$$(A|I) = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 0 & 0 & 1 \end{pmatrix}$$

Carry out now operations involving linear row combinations and permutations to bring the left side to  $I$

$$\begin{aligned} & \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & -2 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & -2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & -3 & -3 & -1 & 1 \end{pmatrix} \sim \\ & \sim \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \sim \\ & \begin{pmatrix} 1 & 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 1 & 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \end{aligned}$$

to obtain

$$A^{-1} = \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{6} \\ 0 & \frac{1}{3} & \frac{1}{6} \\ 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

#### 4. Determinants

- $A \in \mathbb{R}^{m \times m}$  a square matrix,  $\det(A) \in \mathbb{R}$  is the oriented volume enclosed by the column vectors of  $A$  (a parallelepiped)
- Geometric interpretation of determinants
- Determinant calculation rules
- Algebraic definition of a determinant

DEFINITION. The determinant of a square matrix  $A = (a_1 \dots a_m) \in \mathbb{R}^{m \times m}$  is a real number

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{vmatrix} \in \mathbb{R}$$

giving the (oriented) volume of the parallelepiped spanned by matrix column vectors.

- $m=2$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

- $m=3$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Computation of a determinant with  $m=2$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- Computation of a determinant with  $m = 3$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

- Where do these determinant computation rules come from? Two viewpoints
  - *Geometric viewpoint*: determinants express parallelepiped volumes
  - *Algebraic viewpoint*: determinants are computed from all possible products that can be formed from choosing a factor from each row and each column
- $m = 2$

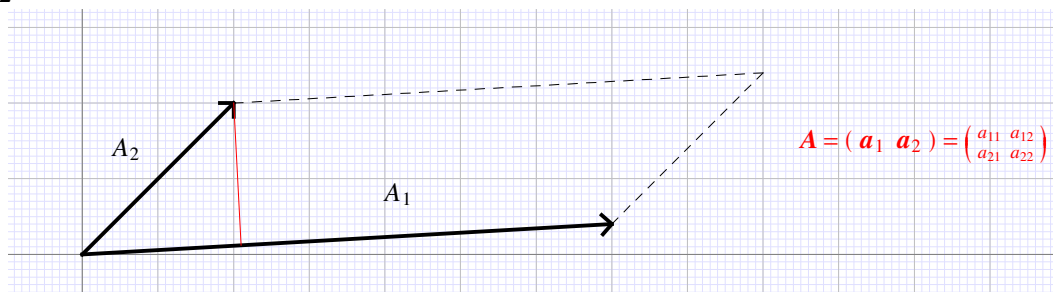


Figure 1.

- In two dimensions a ``parallelepiped'' becomes a parallelogram with area given as

$$(\text{Area}) = (\text{Length of Base}) \times (\text{Length of Height})$$

- Take  $\mathbf{a}_1$  as the base, with length  $b = \|\mathbf{a}_1\|$ . Vector  $\mathbf{a}_1$  is at angle  $\varphi_1$  to  $x_1$ -axis,  $\mathbf{a}_2$  is at angle  $\varphi_2$  to  $x_2$ -axis, and the angle between  $\mathbf{a}_1, \mathbf{a}_2$  is  $\theta = \varphi_2 - \varphi_1$ . The height has length

$$h = \|\mathbf{a}_2\| \sin \theta = \|\mathbf{a}_2\| \sin(\varphi_2 - \varphi_1) = \|\mathbf{a}_2\| (\sin \varphi_2 \cos \varphi_1 - \sin \varphi_1 \cos \varphi_2)$$

- Use  $\cos \varphi_1 = a_{11} / \|\mathbf{a}_1\|$ ,  $\sin \varphi_1 = a_{12} / \|\mathbf{a}_1\|$ ,  $\cos \varphi_2 = a_{21} / \|\mathbf{a}_2\|$ ,  $\sin \varphi_2 = a_{22} / \|\mathbf{a}_2\|$

$$(\text{Area}) = \|\mathbf{a}_1\| \|\mathbf{a}_2\| (\sin \varphi_2 \cos \varphi_1 - \sin \varphi_1 \cos \varphi_2) = a_{11}a_{22} - a_{12}a_{21}$$

- The geometric interpretation of a determinant as an oriented volume is useful in establishing rules for calculation with determinants:

- Determinant of matrix with repeated columns is zero (since two edges of the parallelepiped are identical). Example for  $m = 3$

$$\Delta = \begin{vmatrix} a & a & u \\ b & b & v \\ c & c & w \end{vmatrix} = abw + bcu + cav - abc - vca - wab = 0$$

This is more easily seen using the column notation

$$\Delta = \det(\mathbf{a}_1 \ \mathbf{a}_1 \ \mathbf{a}_3 \ \dots) = 0$$

- Determinant of matrix with linearly dependent columns is zero (since one edge lies in the 'hyperplane' formed by all the others)

- Separating sums in a column (similar for rows)

$$\det(\mathbf{a}_1 + \mathbf{b}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_m) = \det(\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_m) + \det(\mathbf{b}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_m)$$

with  $\mathbf{a}_i, \mathbf{b}_1 \in \mathbb{R}^m$

- Scalar product in a column (similar for rows)

$$\det(\alpha \mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_m) = \alpha \det(\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_m)$$

with  $\alpha \in \mathbb{R}$

- Linear combinations of columns (similar for rows)

$$\det(\mathbf{a}_1 \ \mathbf{a}_2 \ \dots \ \mathbf{a}_m) = \det(\mathbf{a}_1 \ \alpha \mathbf{a}_1 + \mathbf{a}_2 \ \dots \ \mathbf{a}_m)$$

with  $\alpha \in \mathbb{R}$ .

- A determinant of size  $m$  can be expressed as a sum of determinants of size  $m-1$  by expansion along a row or column

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mm} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m2} & a_{m3} & \dots & a_{mm} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m3} & \dots & a_{mm} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m4} & \dots & a_{mm} \end{vmatrix} - \dots + (-1)^{m+1} a_{1m} \begin{vmatrix} a_{21} & a_{23} & \dots & a_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m3} & \dots & a_{m,m-1} \end{vmatrix}$$

- The formal definition of a determinant

$$\det A = \sum_{\sigma \in \Sigma} \nu(\sigma) a_{1i_1} a_{2i_2} \dots a_{mi_m}$$

requires  $mm!$  operations, a number that rapidly increases with  $m$

- A more economical determinant is to use row and column combinations to create zeros and then reduce the size of the determinant, an algorithm reminiscent of Gauss elimination for systems

Example:

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 3 & 10 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 3 & 10 \end{vmatrix} = 20 - 12 = 8$$

The first equality comes from linear combinations of rows, i.e. row 1 is added to row 2, and row 1 multiplied by 2 is added to row 3. These linear combinations maintain the value of the determinant. The second equality comes from expansion along the first column

#### 4.1. Cross product

- Consider  $u, v \in \mathbb{R}^3$ . We've introduced the idea of a scalar product

$$u \cdot v = u^T v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

in which from two vectors one obtains a scalar

- We've also introduced the idea of an exterior product

$$u v^T = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} (v_1 \ v_2 \ v_3) = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}$$

in which a matrix is obtained from two vectors



- Another product of two vectors is also useful, the cross product, most conveniently expressed in determinant-like form

$$u \times v = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - v_2u_3)\mathbf{e}_1 + (u_3v_1 - v_3u_1)\mathbf{e}_2 + (u_1v_2 - v_1u_2)\mathbf{e}_3$$