- **First-order differentiation matrix**. Accurate eigenvalue approximations can be obtained even when continuum boundary conditions are not exactly represented in the discrete formulation. Repeat the above discretization $x_k = kh$, $k = 1, \ldots, m$, $h = 1/(m+1)$ for the first-order differentiation operator ∂_x with eigenvalues ξ associated with eigenfunctions $e^{\xi x}$

$$\partial_x e^{\xi x} = \xi e^{\xi x}.$$

The derivative may be approximated by

$$u_k' = (\partial_x e^{\xi x})_k \cong \frac{e^{\xi(x_k+h)} - e^{\xi(x_k-h)}}{2h} = \frac{e^{\xi h} - e^{-\xi h}}{2h} e^{\xi k h} = \frac{1}{h} \sinh(\xi h)\, e^{\xi k h} = \frac{1}{h} \sinh(\xi h)\, u_k. \tag{1}$$

The eigenproblem

$$\boldsymbol{u}' = \boldsymbol{D}\boldsymbol{u}, \boldsymbol{D} = \frac{1}{2h} \operatorname{diag}([\,-1 \;\; 0 \;\; 1\,]) \in \mathbb{R}^{m \times m},$$

differs from the discretization (1)

$$\boldsymbol{u}' = \boldsymbol{D}\boldsymbol{u} + \boldsymbol{b}, b_1 = -\frac{u_0}{2h}, b_m = \frac{u_{m+1}}{2h},$$

hence $\sin(\xi h)/h$

Figure 1. Comparison of eigenvalues of second-order differentiation matrix $\boldsymbol{D}$ (blue circles) with those