1. Interpolation error

As mentioned, a polynomial interpolant of $f:\mathbb{R}\to\mathbb{R}$ already incorporates the function values $y_i=f(x_i), i=0,\ldots,n$, so additional information on f is required to estimate the error

$$e(t)=f(t)-p(t),$$

when t is not one of the sample points. One approach is to assume that f is smooth, $f\in C^{\infty}(\mathbb{R})$, in which case the error is given by

$$f(t)-p(t)=\frac{f^{(n+1)}(\xi_t)}{(n+1)!}\prod_{i=0}^{n}(t-x_i)=\frac{f^{(n+1)}(\xi_t)}{(n+1)!}w(t), \tag{1}$$

for some $\xi_t\in[x_0,x_n]$, assuming $x_0<x_1<\cdots<x_n$. The above error estimate is obtained by repeated application of Rolle's theorem to the function

$$\Phi(u)=f(u)-p(u)-\frac{f(t)-p(t)}{w(t)}w(u),$$

that has $n+1$ roots at $t,x_0,x_1,\ldots,x_n$, hence its $(n+1)$-order derivative must have a root in the interval (x_0,x_n), denoted by ξ_t

$$\Phi^{(n+1)}(\xi_t)=\frac{\mathrm{d}^{n+1}\Phi}{\mathrm{d}u^{n+1}}(\xi_t)=0=f^{(n+1)}(\xi_t)-\frac{f(t)-p(t)}{w(t)}(n+1)!,$$

establishing (1). Though the error estimate seems promising due to the $(n+1)!$ in the denominator, the derivative $f^{(n+1)}$ can be arbitrarily large even for a smooth function. This is the behavior that arises in the Runge function $f(t)=1/[1+(5t)^2]$ (Fig. 1), for which a typical higher-order derivative appears as

- $f^{(10)}=\dfrac{35437500000000\,(107421875\,t^{10}-64453125\,t^{8}+7218750\,t^{6}-206250\,t^{4}+1375\,t^{2}-1)}{(25\,t^2+1)^{11}}, \|f^{(10)}\|_{\infty}\cong 3.5\times 10^{13}.$

The behavior might be attributable to the presence of poles of f in the complex plane at $t_{1,2}=\pm i/5$, but the interpolant of the holomorphic function $g(t)=\exp(-(5t)^2)$, with a similar power series to f,

- $$f(t)\cong 1-25\,t^2+625\,t^4-15625\,t^6+O(t^7),$$
  $$g(t)\cong 1-25\,t^2+\frac{625\,t^4}{2}-\frac{15625\,t^6}{6}+O(t^7),$$

also exhibits large errors (Fig. 1), and also has a high-order derivative of large norm $\|g\|_{\infty}\cong 3\times 10^{11}$.

- $g^{(10)}(t)=1562500000\,e^{-25t^2}(62500000\,t^{10}-56250000\,t^{8}+15750000\,t^{6}-1575000\,t^{4}+47250\,t^{2}-189),$

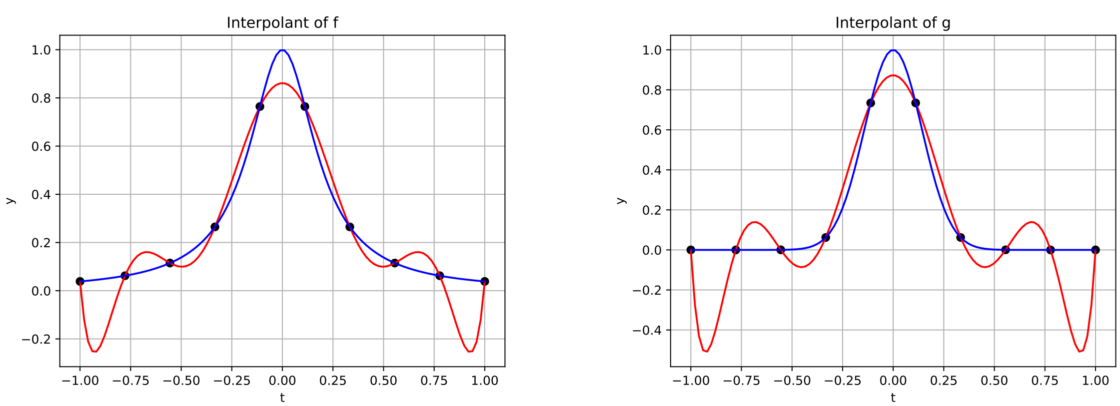
- 

Figure 1. Interpolants of $f(t)=1/[1+(5t)^2]$, $g(t)==\exp(-(5t)^2)$, based on equidistant sample points.

1.1. Error minimization - Chebyshev polynomials

Since $\|f^{(n+1)}\|_\infty$ is problem-specific, the remaining avenue to error control suggested by formula (1) is a favorable choice of the sample points x_i, $i=0,\ldots,n$. The $w(t)$ polynomial

$$w(t)=\prod_{i=0}^{n}(t-x_i)$$

is monic (coefficient of highest power is unity), and any interval $[a,b]\subset\mathbb{R}$ can be mapped to the $[-1,1]$ interval by $t=2(s-a)/(b-a)-1$, leading to consideration of the problem

$$\min_{\boldsymbol{x}}\|w\|_\infty=\min_{\boldsymbol{x}}\max_{-1\leqslant t\leqslant 1}|w(t)|,$$

i.e., finding the sample points or roots of $w(t)$ that lead to the smallest possible inf-norm of $w(t)$. Plots of the Lagrange basis (L18, Fig. 2), or Legendre basis, suggest study of basis functions that have distinct roots in the interval $[-1,1]$ and attain the same maximum. The trigonometric functions satisfy these criteria, and can be used to construct the Chebyshev family of polynomials through

$$T_n(x)=\cos[n\cos^{-1}x]=\cos(n\theta),\cos\theta=x,\theta=\cos^{-1}x.$$

The trigonometric identity

$$\cos[(n+1)\theta]+\cos[(n-1)\theta]=2\cos\theta\cos(n\theta)$$

leads to a recurrence relation for the Chebyshev polynomials

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x),T_0(x)=1,T_1(x)=x.$$

The definition in terms of the cosine function easily leads to the roots, $T_n(x_i)=0$,

$$\cos[n\theta]=0\Rightarrow n\theta_i=(2i-1)\frac{\pi}{2}\Rightarrow\theta_i=\frac{2i-1}{2n}\pi\Rightarrow x_i=\cos\left[\frac{2i-1}{2n}\pi\right],i=1,\ldots,n,$$

and extrema x_j, $T_n(x_j)=(-1)^j$

$$\cos[n\theta]=\pm1\Rightarrow n\theta_j=j\pi\Rightarrow x_j=\cos\left[\frac{j\pi}{n}\right],j=0,1,\ldots,n.$$

The Chebyshev polynomials are not themselves monic, but can readily be rescaled through

$$P_n(x)=2^{1-n}T_n(x),n>0,P_0(x)=1.$$

Since $|T_n(x)|=|\cos(n\theta)|$, the above suggests that the monic polynomials P_n have $\|P_n\|_\infty=2^{1-n}$, small for large n, and are indeed among all possible monic polynomials defined on $[-1,1]$ the ones with the smallest inf-norm.

THEOREM. *The monic polynomial $p:[-1,1]\to\mathbb{R}$ has a inf-norm lower bound*

$$\|p\|_\infty=\max_{-1\leqslant t\leqslant 1}|p(t)|\geqslant 2^{1-n}.$$

Proof. *By contradiction, assume the monic polynomial $p:[-1,1]\to\mathbb{R}$ has $\|q\|_\infty<2^{1-n}$. Construct a comparison with the Chebyshev polynomials by evaluating p at the extrema $x_j=\cos(j\pi/n)$,*

$$(-1)^jp(x_j)\leqslant|p(x_j)|<2^{1-n}=(-1)^jP_n(x_j)=(-1)^j2^{1-n}T_n(x_j).$$

Since the above states $(-1)^jp(x_j)<(-1)^jP_n(x_j)$ deduce

$$(-1)^j[p(x_j)-P_n(x_j)]<0,\text{for } j=0,1,\ldots,n \tag{2}$$

However, p,P_n both monic implies that $p(x_j)-P_n(x_j)$ is a polynomial of degree $n-1$ that would change signs $n+1$ times to satisfy (2), and thus have n roots contradicting the fundamental theorem of algebra. □

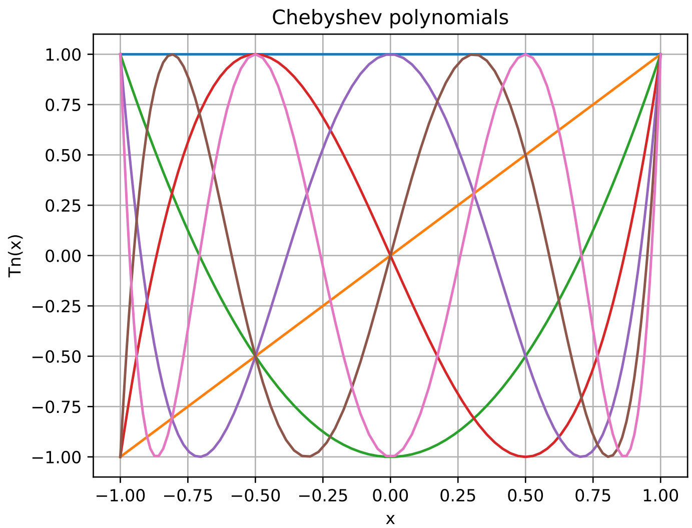


Figure 2. First $n=6$ Chebyshev polynomials

1.2. Best polynomial approximant

Based on the above, the optimal choice of $n+1$ sample points is given by the roots $x_j=\cos(\theta_j)$ of the Chebyshev polynomial of $(n+1)^{\text{th}}$ degree $T_{n+1}(x)$, for which $\cos[(n+1)\theta]=0$,

$$x_j=\cos\left[\frac{\pi}{n+1}\left(\frac{1}{2}+j\right)\right], j=0,\ldots,n,$$

For this choice of sample points the interpolation error has the bound

$$|f(t)-p_n(t)|=\left|\frac{f^{(n+1)}(\xi_t)}{(n+1)!}\prod_{i=0}^{n}(t-x_i)\right|\leqslant\frac{|f^{(n+1)}(\xi_t)|}{(n+1)!}\|P_{n+1}\|_\infty\leqslant\frac{\|f^{(n+1)}\|_\infty}{(n+1)!2^n}.$$