

## LECTURE 35: NONLINEAR OPERATORS

### 1. Advection equation

In late nineteenth century, telegrapher's equations, a system of linear PDEs for current  $I(x, t)$  and voltage  $V(x, t)$

$$\begin{aligned}\frac{\partial}{\partial x}V(x, t) &= -L \frac{\partial}{\partial t}I(x, t) - RI(x, t) \\ \frac{\partial}{\partial x}I(x, t) &= -C \frac{\partial}{\partial t}V(x, t) - GV(x, t)\end{aligned}$$

Heaviside avoided solution of the PDEs by reduction to an algebraic formulation [historical formulation](#), e.g., for the ODE for  $y(t)$

$$\frac{dy}{dt} + ay = b$$

Heaviside considered the associated algebraic problem for  $Y(s)$

$$sY + aY = b \Rightarrow Y(s) = \frac{b}{a+s} \Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)]$$

Why should I refuse a good dinner simply because I don't understand the digestive processes involved? (Heaviside, ?)

Heaviside's formal framework (1890's) for solving ODEs was discounted since it lacked mathematical rigour.

- Russian mathematician 1920's established first results (Vladimirov)
- Theory of Distributions (Schwartz, 1950s)

### 2. Convection equation

Consider function  $f: \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \gg 1$  assumed large,  $f$  of unknown form, difficult to compute for general input. Seek  $g: \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $T: \mathbb{R}^d \rightarrow \mathbb{R}^n$  such that

$$\|f - g \circ T\| < \varepsilon$$

for some  $\varepsilon > 0$ .

Choose a basis set (Monomials, Exponentials, Wavelets)  $\{\phi_1, \phi_2, \dots\}$  to approximation of  $L^2(\mathbb{R})$  functions in Hilbert space

$$g_n(t) = \sum_{j=1}^n (f, \phi_j) \phi_j = \sum_{j=1}^n c_j \phi_j$$

The approximation is convergent if

$$\lim_{n \rightarrow \infty} \|f - g \circ T\| = 0,$$

This assumes  $c_j = (f, \phi_j)$  rapidly decrease.

**THEOREM.** (Parseval) *The Fourier transform is unitary. For  $A, B: \mathbb{R} \rightarrow \mathbb{C}$ , square integrable,  $2\pi$ -periodic with Fourier series*

$$\begin{aligned}A(t) &= \sum_{n=-\infty}^{\infty} a_n e^{int}, B(t) = \sum_{n=-\infty}^{\infty} b_n e^{int}, \\ \sum_{n=-\infty}^{\infty} a_n \bar{b}_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} A(t) \bar{B}(t) dt.\end{aligned}$$

Bessel inequality:

$$\sum_{j=1}^n |(f, \phi_j)|^2 \leq \|f\|_2^2.$$

Fourier coefficient decay: for  $f \in C^{(k-1)}(\mathbb{R})$ ,  $f^{(k-1)}$  absolutely continuous,

$$|c_n| \leq \min_{0 \leq j \leq k} \frac{\|f^{(j)}\|_1}{|n|^j}.$$

In practice: coefficients decay as

- $1/n$  for functions with discontinuities on a set of Lebesgue measure 0;
- $1/n^2$  for functions with discontinuous first derivative on a set of Lebesgue measure 0;
- $1/n^3$  for functions with discontinuous second derivative on a set of Lebesgue measure 0.

Fourier coefficients for analytic functions decay faster than any monomial power  $c_n = o(n^{-p})$ ,  $\forall p \in \mathbb{N}$ , a property known as exponential convergence.

Denote such approximations by  $\mathcal{L}$ , and they are linear

$$\mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L}(f) + \beta \mathcal{L}(g)$$

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Choose a basis set (Monomials, Exponentials, Wavelets)  $\{\phi_1, \phi_2, \dots\}$  to approximation of  $L^2(\mathbb{R})$  functions in Hilbert space

$$g_n(t) = \sum_{j=1}^n c_j \phi_j$$

Let  $\Phi_n = \{\varphi_{k(1)}, \varphi_{k(2)}, \dots, \varphi_{k(n)}\}$  such

$$(f, \varphi_{k(1)}) \geq (f, \varphi_{k(2)}) \geq \dots \geq (f, \varphi_{k(n)}).$$

Choose  $c_j = (f, \varphi_{k(j)})$ , and

$$g_n(t) = \sum_{j=1}^n c_j \phi_j.$$

Denote such approximations by  $\mathcal{G}$ , and they are non-linear.

### 3. Discontinuous solutions

Consider function  $f: \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \gg 1$  assumed large,  $f$  of unknown form, difficult to compute for general input. Seek  $g: \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $T: \mathbb{R}^d \rightarrow \mathbb{R}^n$  such that

$$\|f - g \circ T\| < \varepsilon$$

for some  $\varepsilon > 0$ .

What questions do you ask?

**Does  $T$  exist?**  $\forall f, \varepsilon, \exists T$ , such that  $\|f - g \circ T\| < \varepsilon$

**Can arbitrary  $\varepsilon$  be achieved?**

**Can we construct  $T$ ?**

→ By what procedure?

$$T = T_1 \circ T_2 \circ \dots \circ T_J$$

with  $T_i$  simple modifications of identity (ReLU)

$$\min_{T_1, \dots, T_J} \|f - g \circ T_1 \circ T_2 \circ \dots \circ T_J\|$$

$$T_j(\mathbf{x}) = \eta(\mathbf{A}_j \mathbf{x} + \mathbf{b}_j)$$

$$\eta(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

→ At what cost?

**How big is  $n$ ?**

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