Vandermonde matrix

$$\boldsymbol{M} = \boldsymbol{L}\boldsymbol{U}$$

Mathematica

```
In[29]:= V[n_]:=Table[Subscript[x, i]^j,{i, 0, n},{j, 0, n}];
         n=3; n1=n+1; M=V[n]
```

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{pmatrix}$$

```
In[30]:= LU=Simplify[LUDecomposition[M]][[1]]
```

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 - x_0 & x_1^2 - x_0^2 & x_1^3 - x_0^3 \\ 1 & \frac{x_0 - x_2}{x_0 - x_1} & (x_0 - x_2)(x_1 - x_2) & (x_0 - x_2)(x_1 - x_2)(x_0 + x_1 + x_2) \\ 1 & \frac{x_0 - x_3}{x_0 - x_1} & \frac{(x_0 - x_3)(x_3 - x_1)}{(x_0 - x_2)(x_2 - x_1)} & -((x_0 - x_3)(x_3 - x_1)(x_3 - x_2)) \end{pmatrix}$$

```
In[31]:= L=Table[ If[i<j,LU[[j,i]], If[i==j, 1, 0]],{j,1,n1},{i,1,n1}]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \frac{x_0 - x_2}{x_0 - x_1} & 1 & 0 \\ 1 & \frac{x_0 - x_3}{x_0 - x_1} & \frac{(x_0 - x_3)(x_3 - x_1)}{(x_0 - x_2)(x_2 - x_1)} & 1 \end{pmatrix}$$

```
In[32]:= U=Table[ If[i>=j,LU[[j,i]], 0],{j,1,n1},{i,1,n1}]
```

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & x_1 - x_0 & x_1^2 - x_0^2 & x_1^3 - x_0^3 \\ 0 & 0 & (x_0 - x_2)(x_1 - x_2) & (x_0 - x_2)(x_1 - x_2)(x_0 + x_1 + x_2) \\ 0 & 0 & 0 & -((x_0 - x_3)(x_3 - x_1)(x_3 - x_2)) \end{pmatrix}$$

```
In[33]:= Simplify[Expand[L.U]]
```

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{pmatrix}$$

```
In[36]:= Linv=Simplify[Inverse[L]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{x_1-x_2}{x_0-x_1} & \frac{x_2-x_0}{x_0-x_1} & 1 & 0 \\ \frac{(x_1-x_3)(x_3-x_2)}{(x_0-x_1)(x_0-x_2)} & \frac{(x_0-x_3)(x_2-x_3)}{(x_0-x_1)(x_1-x_2)} & \frac{(x_0-x_3)(x_1-x_3)}{(x_0-x_2)(x_2-x_1)} & 1 \end{pmatrix}$$

```
In[37]:= Uinv=Simplify[Inverse[U]]
```

$$\begin{pmatrix} 1 & \frac{x_0}{x_0-x_1} & -\frac{x_0 x_1}{(x_0-x_2)(x_2-x_1)} & \frac{x_0 x_1 x_2}{(x_0-x_3)(x_3-x_1)(x_3-x_2)} \\ 0 & \frac{1}{x_1-x_0} & \frac{x_0+x_1}{(x_0-x_2)(x_2-x_1)} & -\frac{x_1 x_2+x_0(x_1+x_2)}{(x_0-x_3)(x_3-x_1)(x_3-x_2)} \\ 0 & 0 & \frac{1}{(x_0-x_2)(x_1-x_2)} & \frac{x_0+x_1+x_2}{(x_0-x_3)(x_3-x_1)(x_3-x_2)} \\ 0 & 0 & 0 & -\frac{1}{(x_0-x_3)(x_3-x_1)(x_3-x_2)} \end{pmatrix}$$

```
In[43]:= Simplify[{1,x,x^2,x^3}.Uinv]
```

$$\left\{1, \frac{x_0-x}{x_0-x_1}, -\frac{(x-x_0)(x-x_1)}{(x_0-x_2)(x_2-x_1)}, -\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_0-x_3)(x_3-x_1)(x_3-x_2)}\right\}$$

```
In[44]:= Simplify[{1,x,x^2,x^3}.Uinv.Linv]
```

$$\left\{\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}, -\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_0-x_1)(x_1-x_2)(x_1-x_3)}, -\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_0-x_2)(x_2-x_1)(x_2-x_3)}, -\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_0-x_3)(x_3-x_1)(x_3-x_2)}\right\}$$

```
In[45]:=
```

Transpose Vandermonde matrix

$$\boldsymbol{M}^T = (\boldsymbol{L}\boldsymbol{U})^T$$

```
In[2]:= MT=Transpose[M]
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{pmatrix}$$

```
In[7]:= LUT=Simplify[LUDecomposition[MT]][[1]]
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ x_0 & x_1 - x_0 & x_2 - x_0 & x_3 - x_0 \\ x_0^2 & x_0 + x_1 & (x_0 - x_2)(x_1 - x_2) & (x_0 - x_3)(x_1 - x_3) \\ x_0^3 & x_0^2 + x_1 x_0 + x_1^2 & x_0 + x_1 + x_2 & -((x_0 - x_3)(x_3 - x_1)(x_3 - x_2)) \end{pmatrix}$$

```
In[8]:= LT=Table[ If[i<j,LUT[[j,i]], If[i==j, 1, 0]],{j,1,n1},{i,1,n1}]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_0 & 1 & 0 & 0 \\ x_0^2 & x_0 + x_1 & 1 & 0 \\ x_0^3 & x_0^2 + x_1 x_0 + x_1^2 & x_0 + x_1 + x_2 & 1 \end{pmatrix}$$

```
In[9]:= UT=Table[ If[i>=j,LUT[[j,i]], 0],{j,1,n1},{i,1,n1}]
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & x_1 - x_0 & x_2 - x_0 & x_3 - x_0 \\ 0 & 0 & (x_0 - x_2)(x_1 - x_2) & (x_0 - x_3)(x_1 - x_3) \\ 0 & 0 & 0 & -((x_0 - x_3)(x_3 - x_1)(x_3 - x_2)) \end{pmatrix}$$

```
In[12]:= Simplify[Expand[LT.UT]]
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{pmatrix}$$

```
In[13]:= LTinv=Simplify[Inverse[LT]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -x_0 & 1 & 0 & 0 \\ x_0 x_1 & -x_0 - x_1 & 1 & 0 \\ -x_0 x_1 x_2 & x_1 x_2 + x_0 (x_1 + x_2) & -x_0 - x_1 - x_2 & 1 \end{pmatrix}$$

```
In[14]:= Factor /@ Evaluate[LTinv.{1,x,x^2,x^3}]
```

$$\{1, x - x_0, (x - x_0)(x - x_1), (x - x_0)(x - x_1)(x - x_2)\}$$

```
In[15]:=
```

Compare factorizations of M, M^T

```
In[17]:= L
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \frac{x_0 - x_2}{x_0 - x_1} & 1 & 0 \\ 1 & \frac{x_0 - x_3}{x_0 - x_1} & \frac{(x_0 - x_3)(x_3 - x_1)}{(x_0 - x_2)(x_2 - x_1)} & 1 \end{pmatrix}$$

In[18]:= LT

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ x_0 & 1 & 0 & 0 \\ x_0^2 & x_0+x_1 & 1 & 0 \\ x_0^3 & x_0^2+x_1 x_0+x_1^2 & x_0+x_1+x_2 & 1 \end{pmatrix}$$

In[19]:= U

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & x_1-x_0 & x_1^2-x_0^2 & x_1^3-x_0^3 \\ 0 & 0 & (x_0-x_2)(x_1-x_2) & (x_0-x_2)(x_1-x_2)(x_0+x_1+x_2) \\ 0 & 0 & 0 & -((x_0-x_3)(x_3-x_1)(x_3-x_2)) \end{pmatrix}$$

In[20]:= Transpose[U]

$(1, 0, 0, 0; x_0, x_1-x_0, 0, 0; x_0^2, x_1^2-x_0^2, (x_0-x_2)(x_1-x_2), 0; x_0^3, x_1^3-x_0^3, (x_0-x_2)(x_1-x_2)(x_0+x_1+x_2), -((x_0-x_3)(x_3-x_1)(x_3-x_2)))$

In[21]:= L

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \frac{x_0-x_2}{x_0-x_1} & 1 & 0 \\ 1 & \frac{x_0-x_3}{x_0-x_1} & \frac{(x_0-x_3)(x_3-x_1)}{(x_0-x_2)(x_2-x_1)} & 1 \end{pmatrix}$$

In[22]:=

$$\circ \quad \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \frac{x_0-x_2}{x_0-x_1} & 1 & 0 \\ 1 & \frac{x_0-x_3}{x_0-x_1} & \frac{(x_0-x_3)(x_3-x_1)}{(x_0-x_2)(x_2-x_1)} & 1 \end{pmatrix} \begin{pmatrix} 1 & x_0 & x_0^2 & \\ 0 & x_1-x_0 & x_1^2-x_0^2 & x \\ 0 & 0 & (x_0-x_2)(x_1-x_2) & (x_0-x_2)(x_1 \\ 0 & 0 & 0 & -((x_0-x_3)(\end{pmatrix}$$

Mathematica

```
In[22]:= V[n_]:=Table[Subscript[x, i]^j,{i, 0, n},{j, 0, n}];
         n=1; n1=n+1; B=V[n]
```

$$\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix}$$

```
In[24]:= GaussJordan[B_]:=Module[{BI,m=Length[B],I},
           BI = Transpose[Catenate[{Transpose[B],IdentityMatrix[m]}]];
           For[k=1, k<m, k++,
             For[i=k+1, i<=m, i++,
               lik = -BI[[i,k]]/BI[[k,k]];
               For[j=1, j<=2m, j++,
                 BI[[i,j]] = Simplify[BI[[i,j]] + lik BI[[k,j]]]
               ];
             ];
           ];
           For[k=m, k>1, k--,
             For[i=1, i<k, i++,
               lik = -BI[[i,k]]/BI[[k,k]];
               For[j=1, j<=2m, j++,
                 BI[[i,j]] = Simplify[BI[[i,j]] + lik BI[[k,j]]]
               ];
             ];
           ];
           Return[BI]
         ];
         GaussJordan[B]
In[16]:= I=2
```

```
Set::wrsym: Symbol I is Protected.
```
2

```
In[25]:= V[3]
```

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{pmatrix}$$

```
In[26]:= V[2]
```

$$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix}$$

```
In[28]:= Inverse[V[1]]
```

$$\begin{pmatrix} \frac{x_1}{x_1 - x_0} & -\frac{x_0}{x_1 - x_0} \\ -\frac{1}{x_1 - x_0} & \frac{1}{x_1 - x_0} \end{pmatrix}$$

```
In[29]:=
```

- The observations that lead to the Newton basis are also recovered by symbolic computation software that readily carries out the requisite LU calculations, exemplified here for

$n=3$,

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \frac{x_0-x_2}{x_0-x_1} & 1 & 0 \\ 1 & \frac{x_0-x_3}{x_0-x_1} & \frac{(x_0-x_3)(x_3-x_1)}{(x_0-x_2)(x_2-x_1)} & 1 \end{pmatrix} \begin{pmatrix} 1 & x_0 & x_0^2 & x \\ 0 & x_1-x_0 & x_1^2-x_0^2 & x \\ 0 & 0 & (x_0-x_2)(x_1-x_2) & (x_0-x_2)(x_1+ \\ 0 & 0 & 0 & -((x_0-x_3)(\end{pmatrix}$$

```
In[77]:= V[n_]:=Table[Subscript[x, i]^j,{i, 0, n},{j, 0, n}];
         n=3; n1=n+1; M=V[n]; MT=Transpose[V[n]]
```

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ x_0 & x_1 & x_2 & x_3 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{pmatrix}$$

```
In[78]:= LU=Simplify[LUDecomposition[M]][[1]]
```

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1-x_0 & x_1^2-x_0^2 & x_1^3-x_0^3 \\ 1 & \frac{x_0-x_2}{x_0-x_1} & (x_0-x_2)(x_1-x_2) & (x_0-x_2)(x_1-x_2)(x_0+x_1+x_2) \\ 1 & \frac{x_0-x_3}{x_0-x_1} & \frac{(x_0-x_3)(x_3-x_1)}{(x_0-x_2)(x_2-x_1)} & -((x_0-x_3)(x_3-x_1)(x_3-x_2)) \end{pmatrix}$$

```
In[79]:= L=Table[ If[i<j,LU[[j,i]], If[i==j, 1, 0]],{j,1,n1},{i,1,n1}]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \frac{x_0-x_2}{x_0-x_1} & 1 & 0 \\ 1 & \frac{x_0-x_3}{x_0-x_1} & \frac{(x_0-x_3)(x_3-x_1)}{(x_0-x_2)(x_2-x_1)} & 1 \end{pmatrix}$$

```
In[80]:= U=Table[ If[i>=j,LU[[j,i]], 0],{j,1,n1},{i,1,n1}]
```

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & x_1-x_0 & x_1^2-x_0^2 & x_1^3-x_0^3 \\ 0 & 0 & (x_0-x_2)(x_1-x_2) & (x_0-x_2)(x_1-x_2)(x_0+x_1+x_2) \\ 0 & 0 & 0 & -((x_0-x_3)(x_3-x_1)(x_3-x_2)) \end{pmatrix}$$

```
In[81]:= Simplify[Expand[L.U]]
```

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{pmatrix}$$

```
In[59]:= Linv=Simplify[Inverse[L]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -x_0 & 1 & 0 & 0 \\ x_0 x_1 & -x_0 - x_1 & 1 & 0 \\ -x_0 x_1 x_2 & x_1 x_2 + x_0 (x_1 + x_2) & -x_0 - x_1 - x_2 & 1 \end{pmatrix}$$

```
In[74]:= n=Factor /@ Evaluate[Linv.{1,x,x^2,x^3}]
```

$$\{1, x - x_0, (x - x_0)(x - x_1), (x - x_0)(x - x_1)(x - x_2)\}$$

```
In[73]:= Uinv=Simplify[Inverse[U]]
```

$$\begin{pmatrix} 1 & \frac{1}{x_0 - x_1} & \frac{1}{(x_0 - x_1)(x_0 - x_2)} & \frac{1}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \\ 0 & \frac{1}{x_1 - x_0} & -\frac{1}{(x_0 - x_1)(x_1 - x_2)} & -\frac{1}{(x_0 - x_1)(x_1 - x_2)(x_1 - x_3)} \\ 0 & 0 & \frac{1}{(x_0 - x_2)(x_1 - x_2)} & -\frac{1}{(x_0 - x_2)(x_2 - x_1)(x_2 - x_3)} \\ 0 & 0 & 0 & -\frac{1}{(x_0 - x_3)(x_3 - x_1)(x_3 - x_2)} \end{pmatrix}$$

```
In[76]:= Together /@ Evaluate[Uinv . n]
```

$$\left\{\frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}, -\frac{(x - x_0)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_1 - x_2)(x_1 - x_3)},\right.$$
$$\left.-\frac{(x - x_0)(x - x_1)(x - x_3)}{(x_0 - x_2)(x_2 - x_1)(x_2 - x_3)}, -\frac{(x - x_0)(x - x_1)(x - x_2)}{(x_0 - x_3)(x_3 - x_1)(x_3 - x_2)}\right\}$$

```
In[77]:=
```