# MATH661 S02 - Linear combination approximations

**Posted**: 09/06/23
**Due**: 09/13/23, 11:59PM
**Tracks 1 & 2**: 1. **Track 2**: 2.

*This homework introduces the fundamentals of additive approximation techniques in vector spaces. Read and understand the concepts in L04: Linear combinations in* $\mathcal{R}_m$ *and* $\mathcal{C}^0[0, 2\pi)$*, in particular to the example presented in the text and in the code attached to L04: Figure 2. Additional Julia coding constructs are also introduced. Remember to always execute the code snippets within the lecture notes to understand Julia programming techniques.*

1. Approximate $b(t) = t(\pi - t)(2\pi - t)$ on the interval $[0, 2\pi)$ by the following series and study the convergence behavior of the solution. The pseudo-matrix $A(t)$ arising in the approximation is shown in each case

   a) Cosine series

   $$b(t) \cong \sum_{k=0}^{n} x_k \cos(kt),$$

   $$A(t) = [\ 1 \quad \cos(t) \quad \cos(2t) \quad \ldots \quad \cos(nt)\ ], A: \mathbb{R} \to \mathbb{R}^{n+1}.$$

   b) Trigonometric series

   $$b(t) \cong \sum_{k=0}^{n} [x_k \cos(kt) + y_k \sin(kt)],$$

   $$A(t) = [\ 1 \quad \cos(t) \quad \sin(t) \quad \cos(2t) \quad \sin(2t) \quad \ldots \quad \cos(nt) \quad \sin(nt)\ ], A(t): \mathbb{R} \to \mathbb{R}^{2n+1}.$$

   c) Sine series

   $$b(t) \cong \sum_{k=1}^{n} y_k \sin(kt),$$

   $$A(t) = [\ \sin(t) \quad \sin(2t) \quad \ldots \quad \sin(nt)\ ], A: \mathbb{R} \to \mathbb{R}^n.$$

   d) Triangle wave series

   $$b(t) \cong \sum_{k=1}^{n} z_k W_k(t),$$

   $$A(t) = [\ W_1(t) \quad W_2(t) \quad \ldots \quad W_n(t)\ ], A: \mathbb{R} \to \mathbb{R}^n,$$

   with

   $$W_k(t) = 1 - 4\left|\frac{1}{2} - \left[\!\left[\frac{1}{4} + \frac{kt}{2\pi}\right]\!\right]\right|,$$

where $[\![x]\!]$ is the fractional part of $x$, e.g., $[\![1.15]\!] = 0.15$.

**Solution.** Sample $b(t)$ at points $t_j = (j-1)h$, $j = 1, 2, ..., m$, with spacing $h = 2\pi/m$, and let $\boldsymbol{t}$ denote the vector with components $t_j$. Let $p$ denote the number of columns in the $A(t)$ pseudo-matrix, i.e., the number of functions $u_k(t)$ entering into the approximation of $b(t)$ by the linear combination

$$f_p(t) = \sum_{k=1}^{p} c_k u_k(t). \tag{1}$$

Table 1 defines $c_k$, $u_k(t)$ for each of the four cases, with $k = 1, 2, ..., p$. Case $b$ also conforms to this pattern since the presence of both $\cos(kt)$ and $\sin(kt)$ suggests the use of complex numbers

$$b_n(t) = \sum_{k=0}^{n} [x_k \cos(kt) + y_k \sin(kt)] = \frac{1}{2}\sum_{k=0}^{n} [x_k (e^{ikt} + e^{-ikt}) - i y_k (e^{ikt} - e^{-ikt})] \Rightarrow$$

$$b_n(t) = \frac{1}{2}\left[\sum_{k=0}^{n} (x_k - i y_k) e^{ikt} + \sum_{k=0}^{n} (x_k + i y_k) e^{-ikt}\right] = \frac{1}{2}[p_n(t) + q_n(t)].$$

As expected for $b_n(t) \in \mathbb{R}$, $p_n(t), q_n(t) \in \mathbb{C}$ are complex conjugates

$$\bar{p}_n(t) = \sum_{k=0}^{n} \overline{(x_k - i y_k)}\, \overline{e^{ikt}} = \sum_{k=0}^{n} (x_k + i y_k) e^{-ikt} = q_n(t),$$

hence only one need be computed and the expression

$$f_p(t) = 2\,\mathrm{Re}\left\{\sum_{k=1}^{p} c_k e^{-i(k-1)t}\right\}$$

is obtained, indeed conforming to the same pattern as the other cases.

| Case | $c_k$ | $u_k(t)$ |
|---|---|---|
| $a$ | $y_k$ | $\cos((k-1)\,t)$ |
| $b$ | $x_k + i y_k$ | $e^{-i(k-1)t}$ |
| $c$ | $y_k$ | $\sin(kt)$ |
| $d$ | $z_k$ | $W_k(t)$ |

**Table 1.** Coefficient, basis function definitions.

Problem solution consists of the following steps, leading to results in Fig. 1.

- Define $b(t)$ and the basis functions $u_\alpha(k, t)$, $\alpha \in \{a, b, c, d\}$ to be investigated.
- Define a function to compute the error in approximating $b(t)$ by a series with $p$ terms with coefficients determined by sampling at $m$ points

  $$E(m, p) = \| f_p(\boldsymbol{t}) - b(\boldsymbol{t}) \| / \| b(\boldsymbol{t}) \|,$$

  with $b(\boldsymbol{t}), f_p(\boldsymbol{t}) \in \mathbb{R}^m$ vectors obtained by sampling $b(t), f_p(t)$ at $\boldsymbol{t} \in \mathbb{R}^m$. With $\boldsymbol{c} \in \mathbb{R}^p$ the vector with components $c_k$, $f_p(\boldsymbol{t})$ is evaluated by the matrix-vector product

  $$f_p(\boldsymbol{t}) = A(\boldsymbol{t})\,\boldsymbol{c}.$$

- Test the error function by computing the coefficients for $v(t) = \cos(t)$ and $w(t) = \cos(t) + \cos(5t)$ using basis set $u_a(k, t) = \cos(kt)$. We should obtain zero error once enough terms are included.
- Define a function to construct a convergence plot by evaluating $E(m, p)$ on a of range $p$ values $\varepsilon$
- Choose sufficient sample points to resolve the function $b(t)$, say $m = 100$, and construct convergence plots for the four cases

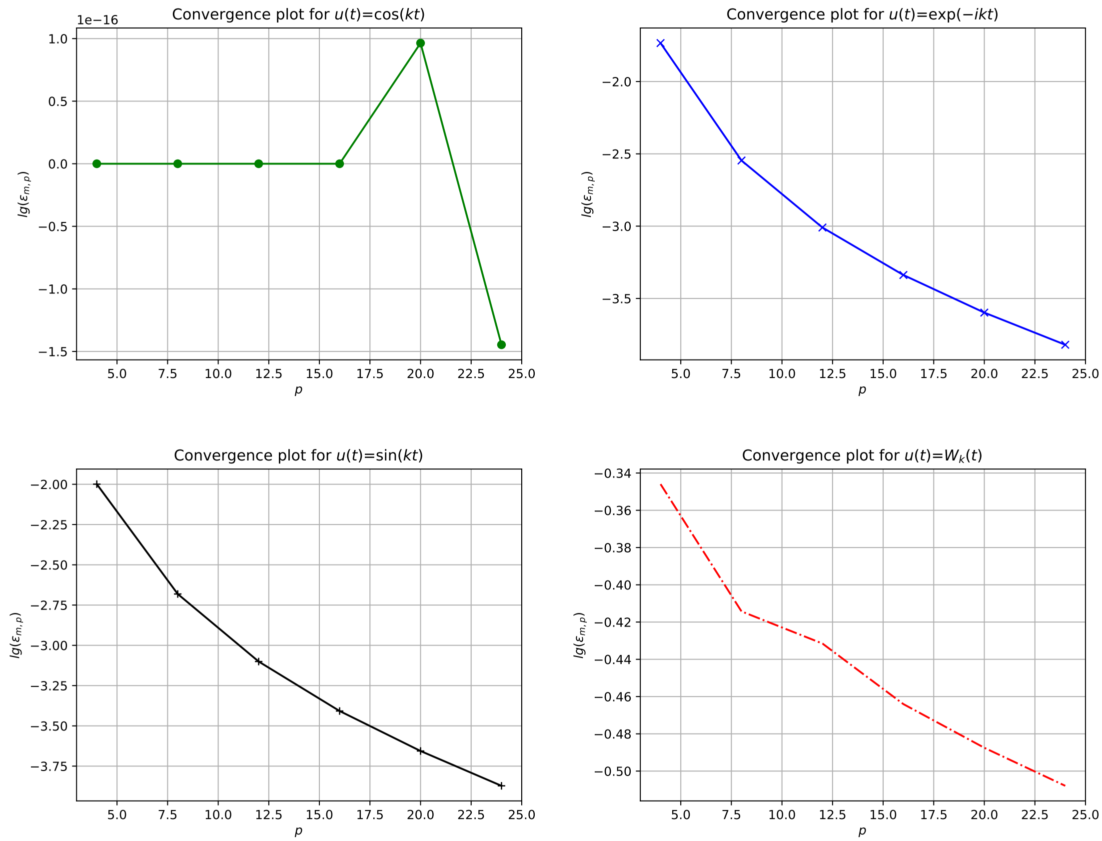


**Figure 1.** Comparison of convergence for different basis sets. No convergence is obtained for $u_a(k,t)=\cos(kt)$ since the basis set is even but $b(t)$ is odd. Convergence is observed for the other basis sets. Faster convergence is obtained for $u_b(k,t)=e^{-ikt}$, $u_c(k,t)=\sin(kt)$, with $\varepsilon\cong 10^{-3}$ (3 significant digits) obtained with $p=11$ terms. Convergence is significantly slower for the triangle functions $u_d(k,t)=W_k(t)$.

2. Study the analytical theory underlying the above approximations by considering the following.

   a) State the convergence theorem for Fourier series and the Fourier coefficient formulas (see, e.g., [1]). Analytically compute the Fourier coefficients for $b(t)=t(\pi-t)(2\pi-t)$. Use of a symbolic computation package (e.g., Maxima, Mathematica) eliminates tedious hand computation.

      **Solution.** For $b:\mathbb{R}\to\mathbb{R}$ with a finite set $\{t_j\}$ of isolated discontinuity points,

$$b(t)=\lim_{n\to\infty}\sum_{k=0}^{n}\ [x_k\cos(kt)+y_k\sin(kt)].$$

      At points of discontinuity

$$\lim_{n\to\infty}\sum_{k=0}^{n}\ [x_k\cos(kt)+y_k\sin(kt)]=\frac{1}{2}[b(t_j^-)+b(t_j^+)].$$

      - The Fourier coefficients are computed by (MATH529: L15)

$$x_k=\frac{1}{\pi}\int_0^{2\pi}b(t)\cos(kt)\,\mathrm{d}t=0,\ y_k=\frac{1}{\pi}\int_0^{2\pi}b(t)\sin(kt)\,\mathrm{d}t=\frac{12}{k^3}.$$

   b) Compare the analytically computed Fourier coefficients with the numerical results obtained in Problem 1, a)-c). Assess the analytically predicted Fourier series convergence by comparison to the numerical results.

      **Solution.** The relevant basis is case $b$, and numerically computed coefficients are compared against analytical results in Fig. 2 .

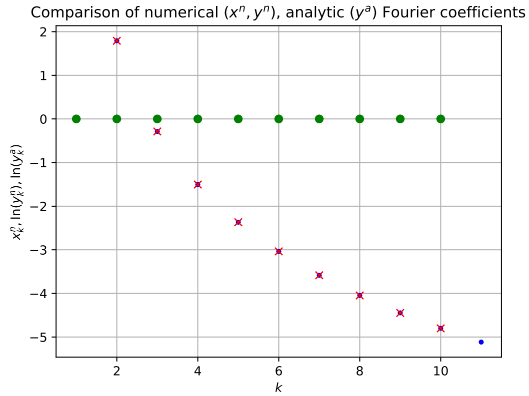


**Figure 2.** Numerical $x_k$ (●), $y_k$ (×), recover analytical $x_k = 0$, $y_k = 12/k^3$ (●).

c) Carry out series approximations as in Problem 1, a)-d) of

$$c(t) = \frac{\pi^3}{4}[H(t) - 2H(t - \pi)]$$

where $H(x)$ is the Heaviside step function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}.$$

**Solution.** The steps of Problem 1 are repeated for $c(t)$ leading to Fig. 3

- Define $H(t), c(t)$
- Sample the function $c(t)$, and construct convergence plots for the four cases

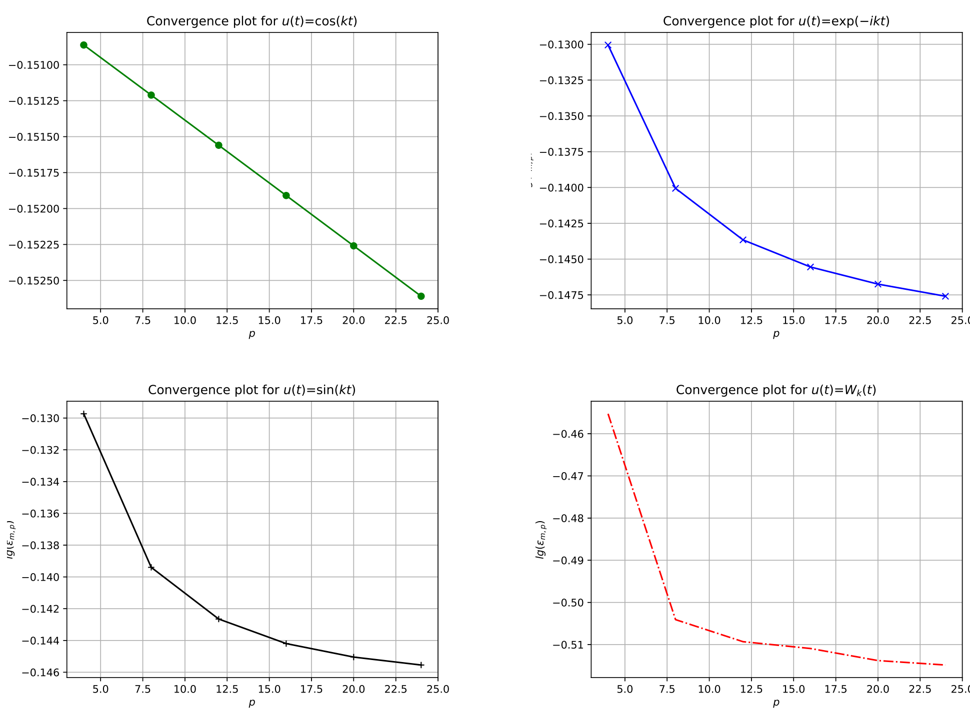


**Figure 3.** Comparison of convergence of approximation of $c(t)$ for different basis sets. Slower convergence is observed by comparison to approximation of $b(t)$. The fastest convergence is observed for the triangle wave basis $W_k(t)$.

d) Again, compare analytically evaluated Fourier coefficients with numerical results. How do the approximations of $c(t)$ behave differently from those of $b(t)$?

**Solution.** Plot $b(t)$ and $c(t)$ to gain insight into different behavior (Fig. 4). Note that $c(t)$ is a discontinuous analogue of $b(t)$.

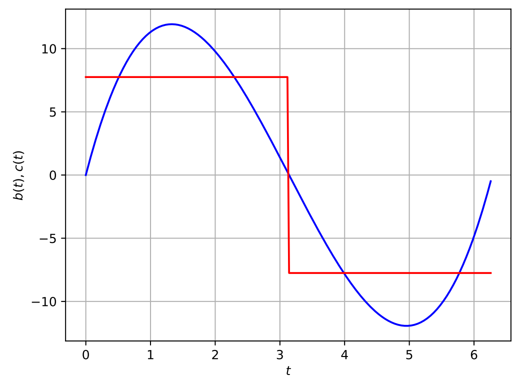


**Figure 4.** $b(t)$ (blue), $c(t)$ (red)

The Fourier coefficients are now

$$x_k = \frac{1}{\pi}\int_0^{2\pi} c(t)\cos(kt)\,\mathrm{d}t = 0,\ y_{2j+1} = \frac{1}{\pi}\int_0^{2\pi} b(t)\sin(kt)\,\mathrm{d}t = \frac{\pi^2}{2j+1},\ y_{2j} = 0.$$

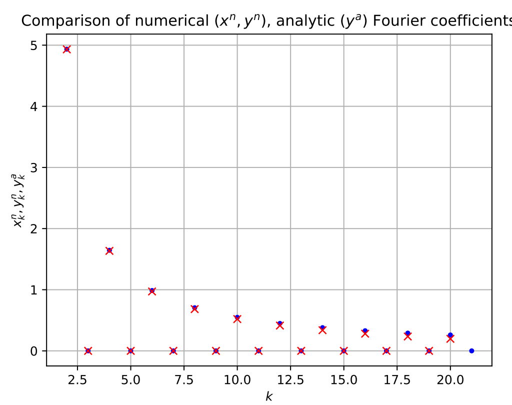


**Figure 5.** Numerical $y_k$ (×), recovers analytical $y_{2j+1} = \pi^2/(2j+1)$ (•).

Convergence is obtained for both $b(t)$ and $c(t)$. Convergence is faster for $b(t)$ in the continuous Fourier basis, while for $c(t)$ it is faster in the discontinuous triangle wave basis. Convergence is slower for the discontinuous function $c(t)$ with coefficient decay $\sim 1/k$ by comparison to the coefficient decay $\sim 1/k^3$ for the continuous function $b(t)$.

# Bibliography


**[1]** Erwin Kreyszig. *Advanced engineering mathematics*. Hoboken, NJ : John Wiley, c2006., 9th ed. edition, 2006.