MATH661 HW05 - Linear algebra analytical practice

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While working on computational aspects in P01, homework will concentrate on analytical properties.

1 Track 1

1. Let λ, μ be distinct eigenvalues of $\boldsymbol{A}$ symmetric, i.e., $\lambda \neq \mu$, $\boldsymbol{A} = \boldsymbol{A}^T$, $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, $\boldsymbol{A}\boldsymbol{y} = \mu\boldsymbol{y}$. Show that $\boldsymbol{x}, \boldsymbol{y}$ are orthogonal.

   Solution. Compute $(\boldsymbol{A}\boldsymbol{x})^T = \boldsymbol{x}^T\boldsymbol{A}^T = \boldsymbol{x}^T\boldsymbol{A} = \lambda\boldsymbol{x}^T \Rightarrow \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{y} = \lambda\boldsymbol{x}^T\boldsymbol{y}$. Multiply $\boldsymbol{A}\boldsymbol{y} = \mu\boldsymbol{y}$ with $\boldsymbol{x}^T$ to obtain $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{y} = \mu\boldsymbol{x}^T\boldsymbol{y}$. Subtracting gives

   $$0 = (\lambda - \mu)\boldsymbol{x}^T\boldsymbol{y} \Rightarrow \boldsymbol{x}^T\boldsymbol{y} = 0$$

   since $\lambda \neq \mu$.

2. Consider

   $$\boldsymbol{A} = \begin{bmatrix} -i & -i & 0 \\ -i & i & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   a) Is $\boldsymbol{A}$ normal?

   b) Is $\boldsymbol{A}$ self-adjoint?

   c) Is $\boldsymbol{A}$ unitary?

   d) Find the eigenvalues and eigenvectors of $\boldsymbol{A}$.

   Solution. a) Observe block structure

   $$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B} & 0 \\ 0 & 1 \end{bmatrix},$$

   such that $\boldsymbol{A}$ is normal if $\boldsymbol{B}$ normal. Note that

   $$\boldsymbol{B}^* = \begin{bmatrix} i & i \\ i & -i \end{bmatrix} = -\boldsymbol{B},$$

   such that $\boldsymbol{B}\boldsymbol{B}^* - \boldsymbol{B}^*\boldsymbol{B} = -\boldsymbol{B}^2 + \boldsymbol{B}^2 = \boldsymbol{0}$, hence $\boldsymbol{B}$ is normal.

   Note: Always look for any special properties of a matrix before attempting calculations.

   b) No, since $\boldsymbol{B}^* = -\boldsymbol{B}$.

   c) No, since $\boldsymbol{B}\boldsymbol{B}^* = -\boldsymbol{B}^2$.

   d) Block structure of $\boldsymbol{A}$ gives $\lambda_3 = 1$, $\boldsymbol{x}_3 = \boldsymbol{e}_3$, with remaining eigenvectors obtained by those of $\boldsymbol{B}$

   $$\boldsymbol{B}\boldsymbol{q} = \mu\boldsymbol{q} \Rightarrow \mu = \pm i\sqrt{2}, \boldsymbol{Q} = \begin{bmatrix} 1 & -1 \\ \sqrt{2} - 1 & \sqrt{2} + 1 \end{bmatrix} \Rightarrow$$

   $$\boldsymbol{X} = \begin{bmatrix} 1 & -1 & 0 \\ \sqrt{2} - 1 & \sqrt{2} + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{\Lambda} = \operatorname{diag}(i\sqrt{2}, -i\sqrt{2}, 1)\,.$$

3. Find the eigenvalues and eigenvectors of the matrix $\boldsymbol{R} \in \mathbb{R}^{3\times 3}$ expressing rotation around the $z-$axis (unit vector $\boldsymbol{e}_3 = [\ 0\ \ 0\ \ 1\ ]^T$).

Solution. The rotation map $\boldsymbol{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is linear

$$\boldsymbol{f}(\alpha \boldsymbol{u} + \beta \boldsymbol{v}) = \alpha \boldsymbol{f}(\boldsymbol{u}) + \beta \boldsymbol{f}(\boldsymbol{v})$$

and the matrix $\boldsymbol{R}$ representing this rotation has columns given by the image of a basis set

$$\boldsymbol{R} = [\ \boldsymbol{f}(\boldsymbol{e}_1)\ \ \boldsymbol{f}(\boldsymbol{e}_2)\ \ \boldsymbol{f}(\boldsymbol{e}_3)\ ] = [\ \cos\theta\, \boldsymbol{e}_1 + \sin\theta \boldsymbol{e}_2\ \ -\sin\theta \boldsymbol{e}_1 + \cos\theta \boldsymbol{e}_2\ \ \boldsymbol{e}_3\ ] = \boldsymbol{I}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

One eigenpair is

$$\lambda_3 = 1, \boldsymbol{x}_3 = \boldsymbol{e}_3,$$

since $\boldsymbol{R}\boldsymbol{e}_3 = \boldsymbol{e}_3$; a rotation does not change a vector along the axis of rotation. The characteristic polynomial $p_3(\lambda) = (\lambda - 1)(\lambda^2 - 2\cos\theta + 1)$ also has roots $\lambda_1 = e^{-i\theta}$, $\lambda_2 = e^{i\theta}$ with associated eigenvectors

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \boldsymbol{e}_1 + i\boldsymbol{e}_2, \boldsymbol{x}_2 = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} = i\boldsymbol{e}_1 + \boldsymbol{e}_2\,.$$

The above can be verified in an insightful manner by recalling that rotation does not change vector lengths (isometric mapping) hence $\boldsymbol{R}$ is orthogonal, and $\boldsymbol{R}^{-1} = \boldsymbol{R}^T$, such that the eigenvalue relation $\boldsymbol{R}\boldsymbol{x} = \lambda \boldsymbol{x}$ leads to

$$\boldsymbol{x} = \lambda \boldsymbol{R}^{-1}\boldsymbol{x} = \lambda \boldsymbol{R}^T \boldsymbol{x}.$$

Verify that $\boldsymbol{x}_1$ is an eigenvector by computing,

$$\lambda_1\, \boldsymbol{R}^T \boldsymbol{x}_1 = e^{-i\theta}\begin{bmatrix} \cos\theta\, \boldsymbol{e}_1^T + \sin\theta \boldsymbol{e}_2^T \\ -\sin\theta \boldsymbol{e}_1^T + \cos\theta \boldsymbol{e}_2^T \\ \boldsymbol{e}_3^T \end{bmatrix}(\boldsymbol{e}_1 + i\boldsymbol{e}_2) = e^{-i\theta}\begin{bmatrix} \cos\theta + i\sin\theta \\ -\sin\theta + i\cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \boldsymbol{x}_1.$$

A similar verification can be done for $\boldsymbol{x}_2$. Note how thinking of $\boldsymbol{R}$ as a collection of column vectors leads to a concise, elegant solution of the eigenvalue problem. Also, notice that for rotation within the $x_1 x_2$ plane the eigenvectors are outside the real $x_1 x_2$ plane and that all eigenvalues are of unit absolute value since the rotation is isometric.

4. Find the eigenvalues and eigenvectors of the matrix $\boldsymbol{S} \in \mathbb{R}^{3\times 3}$ expressing rotation around the axis with unit vector $\boldsymbol{l} = \frac{1}{\sqrt{3}}[\ 1\ \ 1\ \ 1\ ]^T$.

Solution. This is the same as the above problem, but in a different basis, one in which the axis of rotation is $[\ 1\ \ 1\ \ 1\ ]^T$ instead of $[\ 0\ \ 0\ \ 1\ ]^T$. As before, one eigenpair is $\lambda_3 = 1$, with $\boldsymbol{x}_3 = \boldsymbol{l}$ along the rotation axis. Above, the other two eigenvectors were found using unit vectors perpendicular to the rotation axis. One can apply Gram-Schmidt to find $\boldsymbol{j}, \boldsymbol{k}$ orthonormal to $\boldsymbol{l}$ or simply observe that

$$[\ \sqrt{2}\,\boldsymbol{j}\ \ \sqrt{6}\,\boldsymbol{k}\ \ \sqrt{3}\,\boldsymbol{l}\ ] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$$

defines an orthogonal matrix $\boldsymbol{B} = [\ \boldsymbol{j}\ \ \boldsymbol{k}\ \ \boldsymbol{l}\ ]$. Rotating some vector $\boldsymbol{x}$ around the $\boldsymbol{l}$ axis is straightforward in the $\boldsymbol{B}$ basis, so set $\boldsymbol{x} = \boldsymbol{I}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{u} \Rightarrow \boldsymbol{u} = \boldsymbol{B}^T\boldsymbol{x}$. Read this to state: the vector $\boldsymbol{x}$ has coordinates $\boldsymbol{x}$ in the $\boldsymbol{I}$ basis, but coordinates $\boldsymbol{u}$ in the $\boldsymbol{B}$ basis. In the $\boldsymbol{B}$ basis the rotation matrix has columns

$$\boldsymbol{S} = [\ \boldsymbol{f}(\boldsymbol{j})\ \ \boldsymbol{f}(\boldsymbol{k})\ \ \boldsymbol{f}(\boldsymbol{l})\ ] = [\ \cos\theta \boldsymbol{j} + \sin\theta \boldsymbol{k}\ \ -\sin\theta \boldsymbol{j} + \cos\theta \boldsymbol{k}\ \ \boldsymbol{l}\ ] = \boldsymbol{B}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{B}\boldsymbol{R}.$$

The result of the rotation is

$$\boldsymbol{y} = \boldsymbol{S}\boldsymbol{u} = \boldsymbol{B}\boldsymbol{R}\boldsymbol{B}^T\boldsymbol{x},$$

where $\boldsymbol{S}=\boldsymbol{B}\boldsymbol{R}\boldsymbol{B}^T$ is the rotation matrix. The eigendecomposition of $\boldsymbol{R}$ is known from above problem $\boldsymbol{R}=\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^*$ furnishing the eigendecomposition of $\boldsymbol{S}$

$$\boldsymbol{S}=\boldsymbol{B}\boldsymbol{R}\boldsymbol{B}^T=\boldsymbol{B}\boldsymbol{R}\boldsymbol{B}^*=\boldsymbol{B}\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^*\boldsymbol{B}^*=\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^*.$$

The eigenvalues of $\boldsymbol{S}$ are the same of those of $\boldsymbol{R}$ ($e^{\pm i\theta}$,1) and the eigenvectors are

$$\boldsymbol{U}=\boldsymbol{B}\boldsymbol{Q}.$$

5. Compute $\sin(\boldsymbol{A}t)$ for

$$\boldsymbol{A}=\begin{bmatrix} 3 & -9 \\ 2 & -6 \end{bmatrix}.$$

Solution. The matrix is singular hence one eigenvalue is $\lambda_1=0$ with eigenvector

$$\boldsymbol{x}_1=\begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

The other eigenpair is $\lambda_2=-3$,

$$\boldsymbol{x}_2=\begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

and $\boldsymbol{A}$ with distinct eigenvalues is diagonalizable, $\boldsymbol{A}\boldsymbol{X}=\boldsymbol{X}\boldsymbol{\Lambda}\Rightarrow\boldsymbol{A}=\boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1}$. Powers of $\boldsymbol{A}$ are given by

$$\boldsymbol{A}^k=\boldsymbol{X}\boldsymbol{\Lambda}^k\boldsymbol{X}^{-1}.$$

From the Euler formula $e^{i\theta}=\cos\theta+i\sin\theta$ find

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i},$$

which extended to matrix arguments gives

$$\sin(\boldsymbol{A}t)=\frac{1}{2i}[\exp(it\boldsymbol{A})-\exp(-it\boldsymbol{A})].$$

From the power series $\boldsymbol{e}^t=1+t+t^2/2!+\cdots$ obtain

$$\exp(it\boldsymbol{A})=\boldsymbol{I}+it\boldsymbol{A}+(it\boldsymbol{A})^2/2!+\cdots=\boldsymbol{X}\exp(it\boldsymbol{\Lambda})\boldsymbol{X}^{-1},$$

leading to

$$\sin(\boldsymbol{A}t)=\boldsymbol{X}\sin(t\boldsymbol{\Lambda})\boldsymbol{X}^{-1}=\begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -\sin(3t) \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}^{-1}=\begin{bmatrix} -2 & 3 \\ -4/3 & 2 \end{bmatrix}\sin(3t).$$

6. Compute $\cos(\boldsymbol{A}t)$ for

$$\boldsymbol{A}=\begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}.$$

Solution. As above, from eigendecomposition

$$\boldsymbol{A}=\boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1}=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},$$

obtain

$$\cos(\boldsymbol{A}t)=\boldsymbol{X}\cos(\boldsymbol{\Lambda}t)\boldsymbol{X}^{-1}=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \cos(3t) & 0 \\ 0 & \cos(t) \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

7. Compute the SVD of

$$\boldsymbol{A}=\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$$

by finding the eigenvalues and eigenvectors of $\boldsymbol{A}\boldsymbol{A}^T$, $\boldsymbol{A}^T\boldsymbol{A}$.

Solution. With the SVD $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$, the matrix $\boldsymbol{M}=\boldsymbol{A}\boldsymbol{A}^T=\boldsymbol{U}\boldsymbol{\Sigma}^2\boldsymbol{U}^T$ has eigendecomposition

$$\boldsymbol{M}=\frac{1}{\sqrt{10}}\begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 50 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{10}}\begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix},$$

hence

$$\boldsymbol{U}=\frac{1}{\sqrt{10}}\begin{bmatrix} -1 & 3 \\ 3 & 1 \end{bmatrix}, \boldsymbol{\Sigma}=\begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}.$$

From $\boldsymbol{N}=\boldsymbol{A}^T\boldsymbol{A}=\boldsymbol{V}\boldsymbol{\Sigma}^2\boldsymbol{V}^T$ with eigendecomposition

$$\boldsymbol{N}=\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 50 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix},$$

obtain

$$\boldsymbol{V}=\frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}.$$

8. Find the eigenvalues and eigenvectors of $\boldsymbol{A}\in\mathbb{R}^{m\times m}$ with elements $a_{ij}=1$ for all $1\leqslant i,j\leqslant m$. Hint: start with $m=1,2,3$ and generalize.

Solution. Note that $\operatorname{rank}(\boldsymbol{A})=1$ hence $\boldsymbol{A}$ has $\lambda_2=\cdots=\lambda_m=0$ an $m-1$ repeated zero root implying

$$p_{\boldsymbol{A}}(\lambda)=\det(\lambda\boldsymbol{I}-\boldsymbol{A})=\begin{vmatrix} \lambda-1 & -1 & \ldots & -1 \\ -1 & \lambda-1 & \ldots & -1 \\ \vdots & & & \\ -1 & -1 & & \lambda-1 \end{vmatrix}=\lambda^{m-1}(\lambda-\lambda_1).$$

Adding all rows gives a row of $\lambda-m$ such that $\lambda_1=m$ leads to a null determinant with associated eigenvector $\boldsymbol{x}_1=[\,1 \;\; 1 \;\; \ldots \;\; 1\,]^T$. The other eigenvectros are of the form $\boldsymbol{x}_j=[\,1 \;\; 0 \;\; \ldots \;\; -1 \;\; \ldots \;\; 0\,]^T$ with the one in the i^{th} position.

2 Track 2

1. Prove that $\boldsymbol{A}\in\mathbb{C}^{m\times m}$ is normal if and only if it has m orthonormal eigenvectors.

Solution. Apply Schur theorem $\boldsymbol{A}=\boldsymbol{U}\boldsymbol{T}\boldsymbol{U}^*$ such that $\boldsymbol{A}$ normal $\boldsymbol{A}\boldsymbol{A}^*=\boldsymbol{A}^*\boldsymbol{A}$ implies $\boldsymbol{T}\boldsymbol{T}^*=\boldsymbol{T}^*\boldsymbol{T}$, which implies that $\boldsymbol{T}$ is diagonal, as proven by induction

$$\boldsymbol{T}=\begin{bmatrix} t & \boldsymbol{z}^* \\ \boldsymbol{0} & \boldsymbol{S} \end{bmatrix}, \boldsymbol{T}\boldsymbol{T}^*=\begin{bmatrix} t & \boldsymbol{z}^* \\ \boldsymbol{0} & \boldsymbol{S} \end{bmatrix}\begin{bmatrix} \bar{t} & \boldsymbol{0} \\ \boldsymbol{z} & \boldsymbol{S}^* \end{bmatrix}=\begin{bmatrix} t\bar{t}+\boldsymbol{z}^*\boldsymbol{z} & \boldsymbol{z}^*\boldsymbol{S}^* \\ \boldsymbol{S}\boldsymbol{z} & \boldsymbol{S}\boldsymbol{S}^* \end{bmatrix},$$

$$\boldsymbol{T}^*\boldsymbol{T}=\begin{bmatrix} t & \boldsymbol{0} \\ \boldsymbol{z} & \boldsymbol{S}^* \end{bmatrix}\begin{bmatrix} \bar{t} & \boldsymbol{z}^* \\ \boldsymbol{0} & \boldsymbol{S} \end{bmatrix}=\begin{bmatrix} t\bar{t} & t\boldsymbol{z}^* \\ \boldsymbol{z}\bar{t} & \boldsymbol{S}\boldsymbol{S}^* \end{bmatrix},$$

since $t\bar{t}+\boldsymbol{z}^*\boldsymbol{z}=t\bar{t}$ implies $\boldsymbol{z}=\boldsymbol{0}$.

2. Prove that $\boldsymbol{A}\in\mathbb{R}^{m\times m}$ symmetric has a repeated eigenvalue if and only if it commutes with a non-zero skew-symmetric matrix $\boldsymbol{B}$.

Solution. $\boldsymbol{A}$ symmetric is normal and has an orthogonal eigendecomposition $\boldsymbol{A}=\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$, that states that in the $\boldsymbol{Q}$ basis the effect of $\boldsymbol{A}$ is simple scaling of components by the eigenvalues. From $\boldsymbol{B}$ that commutes with $\boldsymbol{A}$, construct $\boldsymbol{C}=\boldsymbol{Q}^T\boldsymbol{B}\boldsymbol{Q}=-\boldsymbol{C}^T$ (to work in $\boldsymbol{Q}$ basis), and find

$$\boldsymbol{\Lambda}\boldsymbol{C}-\boldsymbol{C}\boldsymbol{\Lambda}=\boldsymbol{Q}^T(\boldsymbol{A}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{A})\boldsymbol{Q}=\boldsymbol{0}\Rightarrow\boldsymbol{\Lambda}\boldsymbol{C}=\boldsymbol{C}\boldsymbol{\Lambda}.$$

A repeated eigenvalue is a statement about the components of $\boldsymbol{\Lambda}$. Equality of the (i,j) components of $\boldsymbol{\Lambda}\boldsymbol{C}$ and $\boldsymbol{C}\boldsymbol{\Lambda}$ implies

$$\lambda_i c_{ij}=c_{ij}\lambda_j.$$

From above if there is at least one non-zero c_{ij} element of $\boldsymbol{C}$ then $\lambda_i = \lambda_j$. For the converse choose $c_{ij} = 1 = -c_{ji}$ at i, j for which $\lambda_i = \lambda_j$.

3. Prove that every positive definite matrix $\boldsymbol{K} \in \mathbb{R}^{m \times m}$ has a unique square root $\boldsymbol{B}$, $\boldsymbol{B}$ positive definite and $\boldsymbol{B}^2 = \boldsymbol{K}$.

   Solution. $\boldsymbol{K}$ positive definite implies $\boldsymbol{q}^T \boldsymbol{K} \boldsymbol{q} > 0$ for all $\boldsymbol{q}$ and $\boldsymbol{K} = \boldsymbol{K}^T$. $\boldsymbol{K}$ symmetric admits an orthogonal eigendecomposition $\boldsymbol{K} = \boldsymbol{Q} \boldsymbol{\Lambda} \boldsymbol{Q}^T$ with eigenvalues $\lambda_j > 0$. Define $\boldsymbol{B} = \boldsymbol{Q} \boldsymbol{\Lambda}^{1/2} \boldsymbol{Q}^T$. where diagonal elements of $\boldsymbol{\Lambda}^{1/2}$ are $\sqrt{\lambda_j}$.

4. Find all positive definite orthogonal matrices.

   Solution. In addition to above properties, $\boldsymbol{K} \boldsymbol{K}^T = \boldsymbol{Q} \boldsymbol{\Lambda}^2 \boldsymbol{Q}^T = \boldsymbol{I} \Rightarrow \boldsymbol{\Lambda}^2 = \boldsymbol{I}$, so possible elements of $\boldsymbol{\Lambda}$ are ± 1. Imposing $\boldsymbol{e}_j^T \boldsymbol{Q}^T \boldsymbol{K} \boldsymbol{Q} \boldsymbol{e}_j > 0$ then leads to $\boldsymbol{K} = \boldsymbol{I}$.

5. Find the eigenvalues and eigenvectors of a Householder reflection matrix.

   Solution. Write

   $$\boldsymbol{H} = \boldsymbol{I} - 2 \boldsymbol{q} \boldsymbol{q}^T \in \mathbb{R}^{m \times m}$$

   with $\boldsymbol{q}$ the unit vector normal to the reflection hyperplane and note that $\boldsymbol{H}$ symmetric has an orthogonal eigendecomposition $\boldsymbol{H} = \boldsymbol{U} \boldsymbol{\Lambda} \boldsymbol{U}^T$. Since $\boldsymbol{H}$ is isometric eigenvalues $\lambda = \pm 1$. From

   $$\boldsymbol{H} \boldsymbol{q} = \boldsymbol{q} - 2 \boldsymbol{q} (\boldsymbol{q}^T \boldsymbol{q}) = -\boldsymbol{q}$$

   find eigenpair $(-1, \boldsymbol{q})$. For any $\boldsymbol{v} \perp \boldsymbol{q}$ obtain

   $$\boldsymbol{H} \boldsymbol{v} = \boldsymbol{v}$$

   so $(1, \boldsymbol{v})$ are the remaining $m - 1$ eigenpairs.

6. Find the eigenvalues and eigenvectors of a Givens rotation matrix.

   Solution. The rotation matrix

   $$\boldsymbol{R}(j, k, \theta) = \boldsymbol{I} + (\cos\theta - 1)(\boldsymbol{e}_j \boldsymbol{e}_j^* + \boldsymbol{e}_k \boldsymbol{e}_k^*) - \sin\theta \, (\boldsymbol{e}_j \boldsymbol{e}_k^* - \boldsymbol{e}_k \boldsymbol{e}_j^*)$$

   has $m - 2$ eigenpairs $(\lambda, \boldsymbol{e}_l)$ for $l \neq j$, $l \neq k$. The other eigenpairs are

   $$\lambda_j = e^{-i\theta}, \boldsymbol{x}_j = i \boldsymbol{e}_j + \boldsymbol{e}_k; \lambda_k = e^{i\theta}, \boldsymbol{x}_k = \boldsymbol{e}_j + i \boldsymbol{e}_k.$$

   Verify

   $$\boldsymbol{R} \boldsymbol{x}_j = i \boldsymbol{e}_j + \boldsymbol{e}_k + (\cos\theta - 1)(\boldsymbol{e}_j \boldsymbol{e}_j^* + \boldsymbol{e}_k \boldsymbol{e}_k^*)(i \boldsymbol{e}_j + \boldsymbol{e}_k) - \sin\theta \, (\boldsymbol{e}_j \boldsymbol{e}_k^* - \boldsymbol{e}_k \boldsymbol{e}_j^*)(i \boldsymbol{e}_j + \boldsymbol{e}_k) \Rightarrow$$

   $$\boldsymbol{R} \boldsymbol{x}_j = i \boldsymbol{e}_j + \boldsymbol{e}_k + (\cos\theta - 1)(i \boldsymbol{e}_j + \boldsymbol{e}_k) - \sin\theta \, (\boldsymbol{e}_j - i \boldsymbol{e}_k) = (i \cos\theta - \sin\theta) \boldsymbol{e}_j + (\cos\theta + i \sin\theta) \boldsymbol{e}_k = e^{-i\theta} \boldsymbol{x}_j.$$

7. Prove or state a counterexample: If all eigenvalues of $\boldsymbol{A}$ are zero then $\boldsymbol{A} = \boldsymbol{0}$.

   Solution. Counterexample

   $$\boldsymbol{A} = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right],$$

   a Jordan block with 0 diagonal.

8. Prove: A hermitian matrix is unitarily diagonalizable and its eigenvalues are real.

   Solution. Apply Schur theorem $\boldsymbol{A} = \boldsymbol{Q} \boldsymbol{T} \boldsymbol{Q}^* = \boldsymbol{A}^* = (\boldsymbol{Q} \boldsymbol{T} \boldsymbol{Q}^*)^* \Rightarrow \boldsymbol{Q} (\boldsymbol{T} - \boldsymbol{T}^*) \boldsymbol{Q}^* = \boldsymbol{0} \Rightarrow \boldsymbol{T} = \boldsymbol{T}^*$. Since $\boldsymbol{T}$ triangular is equal to its adjoint it must be diagonal with real elements $\boldsymbol{T} = \boldsymbol{\Lambda}$, $\boldsymbol{A} = \boldsymbol{Q} \boldsymbol{\Lambda} \boldsymbol{Q}^*$.