MATH661 HW07 - Least squares, minimax

Posted: 10/18/23
Due: 10/25/23, 11:59PM

These exercises focus on midterm examination preparation.

1 Track 1

1. Find the polynomial of least degree that interpolates the data $\mathcal{D} = \{(x_i,\, y_i),\, i = 0,\, 1,\, ...,\, n\} = \{(3, 10), (7, 146), (1, 2), (2, 1)\}$.

   Solution. Table of divided differences

i	x_i	y_i	$[y_i, y_{i-1}]$	$[y_i, y_{i-1}, y_{i-2}]$	$[y_i, y_{i-1}, y_{i-2}, y_{i-3}]$
0	3	10	−	−	−
1	7	146	34	−	−
2	1	2	24	5	−
3	2	1	−1	5	0

   ,

   leads to $p_2(t) = 10 + 34(t-3) + 5(t-3)(t-7)$.

2. Find the polynomial of least degree that interpolates the data $\mathcal{D} = \{(x_i,\, y_i,\, y_i'),\, i = 0,\, 1,\, ...,\, n\} = \mathcal{D} = \{(3, 10, 14), (1, 2, -6)\}$.

   Solution. Table of divided differences with repetitions

i	x_i	y_i	$[y_i, y_{i-1}]$	$[y_i, y_{i-1}, y_{i-2}]$	$[y_i, y_{i-1}, y_{i-2}, y_{i-3}]$
0	3	10	−	−	−
0	3	10	$y_0' = 14$	−	−
1	1	2	4	5	−
1	1	2	$y_1' = -6$	5	0

   ,

   leads to $p_2(t) = 10 + 14(t-3) + 5(t-3)^2$.

3. Find the polynomial of least degree that interpolates the data $\mathcal{D} = \{(x_i, y_i, y_i', y_i''), i = 0, 1, ..., n\} = \mathcal{D} = \{(1, 2, -6, 10)\}$.

   Solution. Table of divided differences with repetitions

i	x_i	y_i	$[y_i, y_{i-1}]$	$[y_i, y_{i-1}, y_{i-2}]$	$[y_i, y_{i-1}, y_{i-2}, y_{i-3}]$
0	1	2	−	−	−
0	1	2	$y_0' = -6$	−	−
0	1	2	$y_0' = -6$	$y_0'' = 10$	−

   ,

   leads to $p_2(t) = 2 - 6(t-1) + \frac{10}{2}(t-1)^2$.

4. Find a, b, c, d such that

$$S(x) = \begin{cases} 1 - 2x & x \in (-\infty, -3] \\ a + bx + cx^2 + dx^3 & x \in [-3, 4] \\ 157 - 32x & x \in [4, \infty) \end{cases},$$

   is a natural spline on the $[-3, 4]$ interval.

Solution. Continuity of function conditions:

$$\text{at } x = -3 \Rightarrow 1 - 2(-3) = 7 = a + b\,(-3) + c\,(-3)^2 + d\,(-3)^3$$

$$\text{at } x = 4 \Rightarrow a + b\,(4) + c\,(4)^2 + d\,(4)^3 = 157 - 32(4) = 29$$

Continuity of function derivative conditions:

$$\text{at } x = -3 \Rightarrow -2 = b + 2c\,(-3) + 3d\,(-3)^2$$

$$\text{at } x = 4 \Rightarrow b + 2c\,(4) + 3d\,(4)^2 = -32.$$

Obtain system

$$\begin{cases} a - 3b + 9c - 27d & = & 7 \\ a + 4b + 16c + 64d & = & 29 \\ b - 6c + 27d & = & -2 \\ b + 8c + 48d & = & -32 \end{cases},$$

with solution $\left\{a \to \frac{12763}{343}, b \to \frac{5056}{343}, c \to -\frac{312}{343}, d \to -\frac{282}{343}\right\}$.

5. Apply the Gram-Schmidt algorithm to orthonormalize the function set $\{1, t, t^2\}$ with respect to the scalar product

$$(f, g) = \int_{-1}^{+1} f(t)\, g(t)\, \mathrm{d}t.$$

Solution. Obtain $\varphi_0(t) = 1/(1,1) = 1/\sqrt{2}$. Compute $v_1(t) = t - (t, \varphi_0)\varphi_0 = t$, and bring to unit norm

$$\varphi_1(t) = t/(t,t) = \sqrt{\frac{3}{2}}\, t.$$

Finally, compute

$$v_2(t) = t^2 - (t^2, \varphi_1)\, \varphi_1(t) - (t^2, \varphi_0)\, \varphi_0(t) = t^2 - 1/3.$$

Normalize to obtain

$$\varphi_2(t) = v_2(t)/(v_2, v_2) = \sqrt{\frac{45}{8}}\, (t^2 - 1/3).$$

6. Let $\{\varphi_0(t), \varphi_1(t), \varphi_2(t)\}$ denote the orthonormalized set found above. Find the best 2-norm approximant $g(t) = c_0\varphi_0(t) + c_1\varphi_1(t) + c_2\varphi_2(t)$ of $f(t) = \sin(\pi t/2)$ on the interval $[-1, 1]$.

Solution. Since f is odd and φ_0, φ_2 are even deduce $c_0 = c_2 = 0$. For c_1 compute

$$c_1 = (\varphi_1, f) = \frac{8}{\pi^2}\sqrt{\frac{2}{3}}.$$

7. As above, find the best 2-norm approximant of $f(t) = \cos(\pi t/2)$ on the interval $[-1, 1]$.

Solution. As above, now $c_1 = 0$, and compute

$$c_0 = (\varphi_0, f) = \frac{2\sqrt{2}}{\pi}, c_2 = (\varphi_2, f) = \frac{2\sqrt{10}\,(\pi^2 - 12)}{\pi^3}.$$

8. Find the best approximant $g(t) = \lambda t$ of $f(t) = \sin t$ on the interval $[0, \pi/2]$ in the ∞-norm.

Solution. The error $e(t) = \sin t - \lambda t$ has a local extremum at $e'(\xi) = \cos\xi - \lambda = 0 \Rightarrow \xi = \arccos(\lambda)$, and an endpoint extremum at $t = \pi/2$. Best approximant obtained when $-e(\pi/2) = e(\xi) \Rightarrow$

$$\lambda\pi/2 - 1 = \sin\xi - \lambda\xi \Rightarrow \lambda\pi/2 - 1 = \sqrt{1 - \lambda^2} - \lambda \arccos(\lambda).$$

A numerical procedure can be used to to find λ, the root of the above equation.

2 Track 2

1. In the limit $x_1 \to x_0$ the divided difference

$$f[x_0, x_1] = [y_1, y_0] = \frac{y_1 - y_0}{x_1 - x_0}, y_i = f(x_i), i = 0, 1,$$

has limit $f[x_0, x_1] \to f'(x_0)$. Write and establish the validity of the finite difference form of the product rule $(fg)' = f'g + fg'$.

Solution. Let $y_i = f(x_i)$, $z_i = g(x_i)$, $w_i = (fg)(x_i) = f(x_i)g(x_i) = y_i z_i$. The divided difference of the product is

$$(fg)[x_0, x_1] = [w_1, w_0] = \frac{w_1 - w_0}{x_1 - x_0} = \frac{y_1 z_1 - y_0 z_0}{x_1 - x_0}.$$

The product derivative rule suggests

$$[w_1, w_0] = [y_1, y_0] z_j + y_k [z_1, z_0],$$

leading to

$$y_1 z_1 - y_0 z_0 = (y_1 - y_0) z_j + y_k (z_1 - z_0),$$

and identification $j = 1$, $k = 0$ and product rule

$$[w_1, w_0] = [y_1, y_0] z_1 + y_0 [z_1, z_0],$$

or

$$(fg)[x_0, x_1] = f[x_0, x_1]\, g[x_1] + f[x_0]\, g[x_0, x_1].$$

2. Repeat the above for second order finite differences and $(fg)'' = f''g + 2f'g' + fg''$.

Solution. Generalize above result to

$$(fg)[x_0, x_1, x_2] = f[x_0, x_1, x_2]\, g[x_2] + f[x_0, x_1]\, g[x_1, x_2] + f[x_1, x_2]\, g[x_0, x_1] + f[x_0] g[x_0, x_1, x_2]$$

3. A natural cubic spline has zero curvature at the end points. Prove that of all cubic spline interpolations of data $\mathcal{D} = \{(x_i, y_i = f(x_i)), i = 0, 1, ..., n\}$, the natural spline $S(t)$ curvature two-norm is bounded by the function curvature two-norm

$$\int_{x_0}^{x_n} [S''(t)]^2 \, \mathrm{d}t \leqslant \int_{x_0}^{x_n} [f''(t)]^2 \, \mathrm{d}t.$$

Solution. Consider the Hilbert space $L_2[x_0, x_n] = \{f | f\colon [x_0, x_n] \to \mathbb{R}, \exists M \in \mathbb{R}\ \|f\| \leqslant M\}$ of square integrable functions with scalar product, norm

$$(f, g) = \int_{x_0}^{x_n} f(t) g(t) \, \mathrm{d}t, \|f\|^2 = (f, f).$$

The above statement is rewritten as

$$\|S''\|^2 \leqslant \|f''\|^2.$$

Let $g = f - S$ to obtain

$$\|f''\|^2 = (g'' + S'', g'' + S'') = \|S''\|^2 + \|g''\|^2 + 2(S'', g''),$$

and the requested inequality holds if

$$(S'', g'') \geqslant 0.$$

With S_i: $[x_{i-1}, x_i] \rightarrow \mathbb{R}$, $S_i(t) = a_i(t - x_{i-1})^3 + b_i(t - x_{i-1})^2 + m_i(t - x_{i-1}) + y_{i-1}$, integrate over subintervals

$$\int_{x_0}^{x_n} S''(t)\, g''(t)\, \mathrm{d}t = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} S_i''(t)\, g''(t)\, \mathrm{d}t. \tag{1}$$

Integrate by parts

$$\int_{x_{i-1}}^{x_i} S_i''(t)\, g''(t)\, \mathrm{d}t = [S''(t) g'(t)]_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} S_i'''(t)\, g'(t)\, \mathrm{d}t,$$

note that $S_i'''(t) = c_i$ is constant, and replace in (1) to obtain

$$\int_{x_0}^{x_n} S''(t)\, g''(t)\, \mathrm{d}t = \sum_{i=1}^{n} [S''(x_i) g'(x_i) - S''(x_{i-1}) g'(x_{i-1})] - \sum_{i=1}^{n} c_i \int_{x_{i-1}}^{x_i} g'(t)\, \mathrm{d}t \Rightarrow$$

$$\int_{x_0}^{x_n} S''(t)\, g''(t)\, \mathrm{d}t = [S''(x_1) g'(x_1) - S''(x_0) g'(x_0)] + [S''(x_2) g'(x_2) - S''(x_1) g'(x_1)] + \ldots$$

$$[S''(x_n) g'(x_n) - S''(x_{n-1}) g'(x_{n-1})] - \sum_{i=1}^{n} c_i [g(x_i) - g(x_{i-1})].$$

At nodes $g(x_i) = f(x_i) - S(x_i) = 0$ due to the interpolation condition and terms cancel out giving

$$\int_{x_0}^{x_n} S''(t)\, g''(t)\, \mathrm{d}t = S''(x_n) g'(x_n) - S''(x_0) g'(x_0).$$

For natural end conditions $S''(x_n) = S''(x_0) = 0$, hence

$$(S'', g'') = \int_{x_0}^{x_n} S''(t)\, g''(t)\, \mathrm{d}t = 0 \Rightarrow \|f''\|^2 = \|S''\|^2 + \|g''\|^2 \geqslant \|S''\|^2.$$

Note that this leads to

$$\|f''\|^2 = \|S''\|^2 + \|g''\|^2,$$

a Pythagorean theorem, stating that the projection of f'' onto space of first-degree splines S'' is orthogonal.

4. Find a, b, c, d such that

$$|e(1)| = |e(0)| \Rightarrow |\cosh 1 - a - b| = |1 - a| \Rightarrow$$

$$S(x) = \begin{cases} 1 - 2x & x \in (-\infty, -3] \\ a + bx + cx^2 + dx^3 & x \in [-3, 4] \\ 157 - 32x & x \in [4, \infty) \end{cases},$$

is a natural spline on the $[-3, 4]$ interval.

Solution. See above.

5. Present an analysis of the conditioning of quadratic spline interpolation.

Solution. The problem of quadratic spline interpolation is stated as $(\boldsymbol{x}, \boldsymbol{y}) \rightarrow^f \boldsymbol{s}$, and from L21 the slope system is

$$s_{i-1} + s_i = \frac{2}{h_i}(y_i - y_{i-1}), i = 1, 2, \ldots, n,$$

so the problem is linear with a condition number bounded by that of the above bidiagonal matrix $\boldsymbol{S}$

$$\kappa_f \leqslant \kappa(\boldsymbol{S}), \boldsymbol{S} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{bmatrix}, \boldsymbol{A} = \boldsymbol{S}^T \boldsymbol{S} = \begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & \ddots & \ddots & & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 1 \end{bmatrix}.$$

Carry out a numerical experiment.

```
∴ using SparseArrays
```

```
∴ function S(m)
    spdiagm(0 => ones(m), -1 => ones(m-1))
  end;
```

```
∴ collect(S(8))
```

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 1.0 \end{bmatrix} \tag{2}$$

```
∴ r=10:10:100; cond.(Matrix.(S.(r))) ./ r
```

$$\begin{bmatrix} 1.3232029812481902 \\ 1.3015595822121873 \\ 1.2928867470799783 \\ 1.2882662782812822 \\ 1.2854017106315976 \\ 1.2834532213987946 \\ 1.2820423475124278 \\ 1.2809737095514466 \\ 1.2801363174342515 \\ 1.279462470845778 \end{bmatrix} \tag{3}$$

```
∴ r=100:25:250; cond.(Matrix.(S.(r))) ./ r
```

$$\begin{bmatrix} 1.279462470845778 \\ 1.2782407173662882 \\ 1.2774198940886343 \\ 1.2768304891052986 \\ 1.2763867360563506 \\ 1.2760405866745244 \\ 1.2757630315067576 \end{bmatrix} \tag{4}$$

```
∴ r=250:250:1000; cond.(Matrix.(S.(r))) ./ r
```

$$\begin{bmatrix} 1.2757630315067576 \\ 1.2745070304498798 \\ 1.2740858129880002 \\ 1.273874725330468 \end{bmatrix} \tag{5}$$

The above suggests $\boldsymbol{S}$ has a condition number $\kappa(\boldsymbol{S}) = \mathcal{O}(m)$, $\kappa(\boldsymbol{S}) \cong 1.27m$, a well-conditioned problem.

6. Apply the Gram-Schmidt algorithm to orthonormalize the function set $\{1, t, t^2\}$ with respect to the scalar product

$$(f, g) = \int_{-1}^{+1} \frac{f(t)\, g(t)}{\sqrt{1 - t^2}}\, \mathrm{d}t.$$

Solution. As above, obtain Chebyshev polynomials.

7. Find the best approximant $g(t)=a+bt$ of $f(t)=\sin t$ on the interval $[0,\pi/2]$ in the 2-norm and the ∞-norm.

   Solution. See Midterm2 solution.

8. Prove that best inf-norm approximant of $f\colon[-1,1]\to\mathbb{R}$, $f(t)=\cosh(t)$ by a quadratic polynomial has form $p_2(t)=a+bt^2$, with $b=\cosh 1-1$. Compute a.

   Solution. Since f is even, p_2 is also even, and restrict to $t\in[0,1]$. Equioscillation theorem implies that error $e(t)=\cosh t-(a+bt^2)$ satisfies