MATH661 Midterm Examination

Fall 2014 Semester, October 14, 2014

Instructions. *Answer the following questions. Provide a motivation of your approach and the reasoning underlying successive steps in your solution. Write neatly and avoid erasures. Use scratch paper to sketch out your answer for yourself, and then transcribe your solution to the examination you turn in for grading. Illegible answers are not awarded any credit. Presentation of calculations without mention of the motivation and reasoning are not awarded any credit. Each complete, correct solution to an examination question is awarded 4 course grade points. Your primary goal should be to demonstrate understanding of course topics and skill in precise mathematical formulation and solution procedures.*

1. Construct the spline second-degree polynomial interpolation of data $\mathcal{D} = \{(x_i = i\,h, y_i = f(x_i)), i = 0, \ldots, n\}$, $h = 1/n$, where $f: \mathbb{R} \to \mathbb{R}$, $f \in C^{(2)}(\mathbb{R})$ is a periodic function, $\forall x \in \mathbb{R}$ $f(x) = f(x+1)$. Estimate the number of floating point operations required to compute the spline interpolant.

Solution. The interpolation consists of n branches, p_i: $[x_{i-1}, x_i] \to \mathbb{R}$, of form $p_i(t) = a_i(t - x_{i-1})^2 + b_i(t - x_{i-1}) + y_{i-1}$, thus ensuring $p_i(x_{i-1}) = y_{i-1}$, $i = 1, \ldots, n$. There remain $2n$ coefficients to determine, (a_i, b_i), $i = 1, \ldots, n$. Additional conditions

$$p_i(x_i) = y_i, i = 1, \ldots, n \,(n\, \text{interpolation conditions}) \tag{1}$$

$$p_i'(x_i) = p_{i+1}'(x_i), i = 1, \ldots, n-1 \,(n-1\, \text{smoothness conditions}) \tag{2}$$

A total of $2n - 1$ conditions have been imposed for the $2n$ unknown coefficients (a_i, b_i) $i = 1, \ldots, n$ (1 point).

One additional condition can be imposed to capture periodic behavior of $f \in C^{(2)}(\mathbb{R})$,

$$p_1'(x_0) = p_n'(x_n). \tag{3}$$

It is assumed that the data $\mathcal{D}$ reflects periodic behavior of f, i.e., $y_0 = y_n$ (1 point).

Let $d_i = y_i - y_{i-1}$. Conditions (1,2,3) lead to the linear system

$$a_i h^2 + b_i h = d_i, i = 1, \ldots, n \tag{4}$$

$$2a_i h + b_i = b_{i+1}, i = 1, \ldots, n-1 \tag{5}$$

$$b_1 = 2a_n h + b_n \tag{6}$$

with $2n$ equations for $2n$ unknowns. Elimination of a_i from (4),(5) gives

$$b_i + b_{i+1} = \frac{2d_i}{h}, \text{for } i = 1, \ldots, n-1. \tag{7}$$

Eliminating a_n from (4) when $i=n$ and (6) gives

$$b_1+b_n=\frac{2d_n}{h}. \tag{8}$$

The linear system resulting from (7),(8) is cyclic bidiagonal

$$\begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0\\ 0 & 1 & 1 & \dots & 0 & 0\\ & & & \ddots & & \\ & & & & 1 & 1\\ 1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}\begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_{n-1}\\ b_n \end{pmatrix}=\frac{2}{h}\begin{pmatrix} d_1\\ d_2\\ \vdots\\ d_{n-1}\\ d_n \end{pmatrix} \tag{9}$$

After solving (9), the remaining coefficients a_i result from (5),(6) (1 point). Solving (9) requires n operations for reduction to upper triangular form, n operations for back substitution, with an additional n operations to find coefficients a_i, for a total estimate of $\mathcal{O}(3n)$ floating point operations.

2. Construct the polynomial interpolant of data $\mathcal{D}=\{(-h,\ y_0=f(-h)),\ (0,\ y_1=f(0)),\ (0, y_1'=f'(0)), (h, y_2=f(h))\}$, obtained by sampling a function $f\colon\mathbb{R}-\mathbb{R},\ f\in C^{(3)}(\mathbb{R})$.

Solution. The desired Hermite interpolant can be computed by constructing the divided difference with repetitions table

i	x_i	y_i	$[y_i,y_{i-1}]$	$[y_i,y_{i-1},y_{i-1}]$	$[y_i,y_{i-1},y_{i-2},y_{i-3}]$
0	$-h$	y_0	$-$	$-$	$-$
1	0	y_1	$\frac{y_1-y_0}{h}$	$-$	$-$
1	0	y_1	y_1'	$\frac{hy_1'-y_1+y_0}{h^2}$	$-$
2	h	y_2	$\frac{y_2-y_1}{h}$	$\frac{y_2-y_1-hy_1'}{h^2}$	$\frac{1}{2h^3}(y_2-2hy_1'-y_0)$

(2 points) leading to polynomial

$$p(x)=y_0+\frac{y_1-y_0}{h}(x+h)+\frac{hy_1'-y_1+y_0}{h^2}(x+h)x+\frac{1}{2h^3}(y_2-2hy_1'-y_0)(x+h)x^2.$$

(2 points). Check for data from $f(x)=x^3$, $h=1$, $\mathcal{D}=\{(-1,-1),(0,0),(0,0),(1,1)\}$ gives

$$p(x)=-1+\frac{1}{1}(x+1)-\frac{1}{1}(x+1)x+\frac{2}{2}(x+1)x^2=x^3\checkmark.$$

3. Determine the least squares approximation of data $\mathcal{D}=\{(x_i=ih, y_i=f(x_i)), i=0,1,\dots,m\}$, $h=2\pi/m$, $m\gg 2$, of form

$$p(t)=a_0+a_1\cos(t)+a_2\sin(t),$$

with $f \in C^{\infty}(\mathbb{R})$, a periodic function, $\forall x \in \mathbb{R},\ f(x) = f(x + 2\pi)$.

Solution. Introduce the scalar product

$$(f, g) = \sum_{i=0}^{m} f(x_i)\, g(x_i). \text{ (1 point)}$$

The Wronskian associated with functions $\{1, \cos(t), \sin(t)\}$ is

$$W = \begin{vmatrix} 1 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \\ 0 & -\cos(t) & -\sin(t) \end{vmatrix} = 1 \neq 0,$$

non-zero everywhere, hence $\{1, \cos(t), \sin(t)\}$ are linearly independent.

The least squares approximant of the data is determined by solving the normal equations

$$N a = b$$

with

$$N = B^T B \in \mathbb{R}^{3\times 3}, b = B^T y \in \mathbb{R}^3$$

$$B = \begin{pmatrix} 1 & \cos(x) & \sin(x) \end{pmatrix} \in \mathbb{R}^{(m+1)\times 3},$$

with x, $y \in \mathbb{R}^{m+1}$, column vectors constructed from the data and $\cos(x)$, $\sin(x) \in \mathbb{R}^{m+1}$, column vectors obtained by applying cosine, sine to the data.