January 2013 SciComp #6. Find a two-point Gaussian quadrature for the integral

$$F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t, s > 0.$$

Derive the error expression, its leading order, and how it scales with s as $s \to \infty$.

Solution. (In exam response style). The quadrature rule is of form

$$F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \sum_{i=1}^{2} w_i f(t_i, s) + R(s)$$

with quadrature nodes t_1, t_2 roots of $p_2(t)$, the degree two polynomial from orthogonal family $\{p_0(t), p_1(t), p_2(t)\}$ constructed by Gram-Schmidt orthogonilization with scalar product

$$\langle f, g\rangle = \int_0^\infty e^{-st} f(t) g(t)\,\mathrm{d}t.$$

Construct orthogonal family by applying Gram-Schmidt to $\{1, t, t^2\}$:

Degree 0. $p_0(t) = 1/<1,1> = \left(\int_0^\infty e^{-st}\,\mathrm{d}t\right)^{-1} = \left(-\frac{1}{s}[e^{-st}]_{t=0}^{t\to\infty}\right)^{-1} = \left(-\frac{1}{s}(0-1)\right)^{-1} = s$

Degree 1. $q_1(t) = t - \langle t, s\rangle p_0(t)$. Compute

$$q_1(t) = t - s\int_0^\infty e^{-st}\, t\, s\,\mathrm{d}t = t - s^2 \int_0^\infty e^{-st}\, t\,\mathrm{d}t$$

Integration by parts ($\mathrm{d}v = e^{-st}\,\mathrm{d}t \Rightarrow v = -\frac{1}{s} e^{-st}$, $u = t \Rightarrow \mathrm{d}u = \mathrm{d}t$) gives

$$\int_0^\infty e^{-st}\, t\,\mathrm{d}t = \left[-\frac{t}{s} e^{-st}\right]_{t=0}^{t\to\infty} + \frac{1}{s}\int_0^\infty e^{-st}\,\mathrm{d}t = \frac{1}{s^2},$$

hence

$$q_1(t) = t - 1.$$

Normalize

$$p_1(t) = \frac{q_1(t)}{\langle q_1, q_1\rangle} = \frac{t-1}{\int_0^\infty e^{-st} (t-1)^2\,\mathrm{d}t}$$

Compute

$$\int_0^\infty e^{-st} (t-1)^2\,\mathrm{d}t = \int_0^\infty e^{-st} t^2\,\mathrm{d}t - 2\int_0^\infty e^{-st}\, t\,\mathrm{d}t + \int_0^\infty e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-st} t^2\,\mathrm{d}t - \frac{2}{s^2} + \frac{1}{s}$$

Integration by parts ($\mathrm{d}v = e^{-st}\,\mathrm{d}t \Rightarrow v = -\frac{1}{s} e^{-st}$, $u = t^2 \Rightarrow \mathrm{d}u = 2t\mathrm{d}t$) gives

$$\int_0^\infty e^{-st} t^2\,\mathrm{d}t = \left[-\frac{t^2}{s} e^{-st}\right]_{t=0}^{t\to\infty} + \frac{2}{s}\int_0^\infty e^{-st}\, t\,\mathrm{d}t = \frac{2}{s^3},$$

hence

$$p_1(t) = \frac{t-1}{\frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s}} = \frac{s^3}{2-2s+s^2}(t-1)$$

Degree 2. Compute

$$q_2(t) = t^2 - \langle t^2, p_1 \rangle p_1(t) - \langle t^2, p_0 \rangle p_0(t)$$

$$q_2(t) = t^2 - \frac{s^3(t-1)}{2-2s+s^2} \int_0^\infty e^{-st} t^2 (t-1) \, \mathrm{d}t - s \int_0^\infty e^{-st} t^2 \, \mathrm{d}t$$

Integration by parts ($\mathrm{d}v = e^{-st} \mathrm{d}t \Rightarrow v = -\frac{1}{s} e^{-st}$, $u = t^3 \Rightarrow \mathrm{d}u = 3t^2 \mathrm{d}t$) gives

$$\int_0^\infty e^{-st} t^3 \, \mathrm{d}t = \left[-\frac{t^3}{s} e^{-st} \right]_{t=0}^{t \to \infty} + \frac{3}{s} \int_0^\infty e^{-st} t^2 \, \mathrm{d}t = \frac{6}{s^4},$$

and reusing previous results gives

$$q_2(t) = t^2 - \frac{s^3(t-1)}{2-2s+s^2} \left(\frac{6}{s^4} - \frac{2}{s^3} \right) - s \frac{2}{s^3} = t^2 - \frac{2(3-s)(t-1)}{s(2-2s+s^2)} - \frac{2}{s^2}$$

$$q_2(t) = t^2 - \frac{2(3-s)}{s(2-2s+s^2)} t + \frac{2}{s} \left[\frac{3-s}{2-2s+s^2} - \frac{1}{s} \right] = t^2 - \frac{2(3-s)}{s(2-2s+s^2)} t - \frac{2}{s} \left[\frac{2s^2-5s+2}{s(s^2-2s+2)} \right]$$

Roots of $p_2(t) = q_2(t) / \langle q_2, q_2 \rangle$ are the same as those of q_2

$$t_{1,2} = \frac{(3-s)}{s(2-2s+s^2)} \pm \sqrt{\left[\frac{2(3-s)}{s(2-2s+s^2)} \right]^2 + \frac{2}{s} \left[\frac{2s^2-5s+2}{s(s^2-2s+2)} \right]}$$

The quadrature weights are determined by method of moments,

$$\int_0^\infty e^{-st} \cdot 1 \, \mathrm{d}t = \frac{1}{s} = w_1 + w_2$$

$$\int_0^\infty e^{-st} \cdot t \, \mathrm{d}t = \frac{1}{s^2} = w_1 t_1 + w_2 t_2$$

The quadrature formula is exact for $f \in \Pi_3$ (cubic polynomials), hence Taylor expansion gives

$$F(s) = \int_0^\infty e^{-st} \cdot (T_3(t) + R_3(t)) \, \mathrm{d}t = \sum_{i=1}^{2} w_i f(t_i, s) + \int_0^\infty e^{-st} \cdot R_3(t) \, \mathrm{d}t$$

with $T_3(t)$ the cubic polynomial term from the Taylor series and $R_3(t) = \frac{f^{(\mathrm{iv})}(0)}{4!} t^4$ the (MacLaurin) remainder leading to

$$R(s) = \frac{f^{(\mathrm{iv})}(0)}{4!} \int_0^\infty e^{-st} \cdot t^4 \, \mathrm{d}t.$$

Integration by parts ($\mathrm{d}v = e^{-st} \mathrm{d}t \Rightarrow v = -\frac{1}{s} e^{-st}$, $u = t^4 \Rightarrow \mathrm{d}u = 4t^3 \mathrm{d}t$) gives

$$\int_0^\infty e^{-st} t^4 \, \mathrm{d}t = \left[-\frac{t^4}{s} e^{-st} \right]_{t=0}^{t \to \infty} + \frac{4}{s} \int_0^\infty e^{-st} t^4 \, \mathrm{d}t = \frac{4!}{s^5},$$

so the quadrature error is

$$R(s) = \frac{f^{(\mathrm{iv})}(0)}{s^5} = \mathcal{O}(s^{-5})$$

with fifth-order decay as $s \to \infty$.