MATH661: Homework 1 Solution
Issued: 8/31/15
Due: 9/9/15

Notation: K&C = Kincaid & Cheney textbook
In green: mathematical technique appearing in problem

1.a. K&C p.335, 6.2.2 (Problem approach, continuity $\delta-\varepsilon$ definition, vector space embedding)
Solution. This model solution shows the basic steps you should follow to construct proofs:

1. *State the problem in mathematical terms, specifying the notation.* Let $C(A)$ denote the set of continuous functions with domain A. The problem asks for the proof of:

$$f \in C(\mathbb{R}) \Rightarrow f[x_0, \ldots, x_n] \in C(\mathbb{R}^{n+1}). \tag{1}$$

2. *Apply definitions.* Two definitions arise here: that of continuity (applied twice), and that of divided differences.
   a) By definition of continuity, if $f \in C(\mathbb{R})$ then $\forall x \in \mathbb{R}$, $\forall \varepsilon > 0$, there does exist $\delta > 0$ (δ can depend on ε,x), such that for $\forall y$ satisfying $|x-y| < \delta$, it is true that $|f(x) - f(y)| < \varepsilon$. (one always finds a small enough interval for the argument of a function to bound the variation of the function)
   b) By definition of continuity, $f[x_0, \ldots, x_n] \in C(\mathbb{R}^{n+1})$ if $\forall \boldsymbol{x} = (\begin{array}{ccc} x_0 & \ldots & x_n \end{array})^T \in \mathbb{R}^{n+1}$, $\forall \sigma > 0$, there does exist some $\rho > 0$ (ρ can depend on $\boldsymbol{x}, \sigma$), such that for $\forall \boldsymbol{y} \in \mathbb{R}^{n+1}$ satisfying $\|\boldsymbol{x} - \boldsymbol{y}\| < \rho \Rightarrow |f[x_0, \ldots, x_n] - f[y_0, \ldots, y_n]| < \sigma$.
   c) By definition of divided differences, $f[x_j] = f(x_0 + jh)$ for $j = 0, \ldots, n$, and

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{nh}.$$

3. *Identify a proof strategy.* In this problem, recognize that $f[x_0, \ldots, x_n]$ is a linear combination of $\{f_0, \ldots, f_n\}$, $f_j = f(x_j)$, $j = 0, \ldots, n$. The task at hand is to show that continuity of the *univariate* function f implies continuity of the *multivariate* function $f[x_0, \ldots, x_n]$. This embedding of a lower dimensional entity into a higher dimensional one is the key aspect of the problem. As in all problems where an explicit counting is evident (the role played by the indices $0, 1, \ldots, n$), complete induction is a viable strategy, and leads us to focus on what happens when increasing dimension by a single unit. Consider now $\boldsymbol{x} = (\begin{array}{ccc} x_0 & \ldots & x_n \end{array})^T \in \mathbb{R}^{n+1}$, $\boldsymbol{x}^+ = (\begin{array}{cccc} x_0^+ & \ldots & x_n^+ & x_{n+1}^+ \end{array})^T = (\begin{array}{cc} \boldsymbol{x}^T & 0 \end{array})^T \in \mathbb{R}^{n+2}$, and the two norms $\|\ \|: \mathbb{R}^n \to \mathbb{R}_+$, $\|\ \|^+: \mathbb{R}^{n+1} \to \mathbb{R}_+$. Can we state anything about the norm of the higher dimensional entity, $\|\boldsymbol{x}^+\|^+$ from knowledge of the lower dimensional entity $\|\boldsymbol{x}\|$? Yes, for any $p-$norm

$$\|\boldsymbol{x}^+\|_p^+ = \left(\sum_{i=0}^{n+1} |x_i^+| \right)^{1/p} = \left(\sum_{i=0}^{n} |x_i| \right)^{1/p} = \|\boldsymbol{x}\|_p, \tag{2}$$

   or, in words, extending a vector by a zero component preserves the lower dimensional p-norm value (Question: is this true for any norm?). In particular this holds true for the $p \to \infty$, inf-norm $\|\boldsymbol{x}\| = \max_{0 \leqslant i \leqslant n} |x_i|$.

4. *Apply your proof strategy.* Denote (1) by $\mathcal{P}(n)$, and prove by complete induction:
   i. $\mathcal{P}(0)$: $f \in C(\mathbb{R}) \Rightarrow f[x_0] = f(x_0) \in C(\mathbb{R})$ is a case of $A \Rightarrow A$ (always true)
   ii. Assume $\mathcal{P}(n)$ true, hence when $\|\boldsymbol{x}_0 - \boldsymbol{y}_0\| < \rho_0 \Rightarrow |f[x_0, \ldots, x_n] - f[y_0, \ldots, y_n]| < (n+1)h\sigma/2$, and when $\|\boldsymbol{x}_1 - \boldsymbol{y}_1\| < \rho_1 \Rightarrow |f[x_1, \ldots, x_{n+1}] - f[y_1, \ldots, y_{n+1}]| < (n+1)h\sigma/2$, $\boldsymbol{x}_0 = (\begin{array}{ccc} x_0 & \ldots & x_n \end{array})$, $\boldsymbol{x}_1 = (\begin{array}{ccc} x_1 & \ldots & x_{n+1} \end{array})$, $\boldsymbol{y}_0 = (\begin{array}{ccc} y_0 & \ldots & y_n \end{array})$, $\boldsymbol{y}_1 = (\begin{array}{ccc} y_1 & \ldots & y_{n+1} \end{array}) \in \mathbb{R}^{n+1}$.
   iii. Establish $\mathcal{P}(n+1)$. Consider $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n+2}$, $\boldsymbol{x} = (\begin{array}{ccc} x_0 & \ldots & x_{n+1} \end{array})^T = (\begin{array}{cc} \boldsymbol{x}_0^T & x_{n+1} \end{array})^T = (\begin{array}{cc} x_0 & \boldsymbol{x}_1^T \end{array})^T$, $\boldsymbol{y} = (\begin{array}{ccc} y_0 & \ldots & y_{n+1} \end{array})^T = (\begin{array}{cc} \boldsymbol{y}_0^T & y_{n+1} \end{array})^T = (\begin{array}{cc} y_0 & \boldsymbol{y}_1^T \end{array})^T$. By definition of $\boldsymbol{x}, \boldsymbol{y}$ and norm triangle inequality,

$$\|\boldsymbol{x} - \boldsymbol{y}\| = \left\| \left(\begin{array}{c} \boldsymbol{x}_0 \\ 0 \end{array} \right) - \left(\begin{array}{c} \boldsymbol{y}_0 \\ 0 \end{array} \right) + \left(\begin{array}{c} \boldsymbol{0} \\ x_{n+1} \end{array} \right) - \left(\begin{array}{c} \boldsymbol{0} \\ y_{n+1} \end{array} \right) \right\| \leqslant \left\| \left(\begin{array}{c} \boldsymbol{x}_0 \\ 0 \end{array} \right) - \left(\begin{array}{c} \boldsymbol{y}_0 \\ 0 \end{array} \right) \right\| + \left\| \left(\begin{array}{c} \boldsymbol{0} \\ x_{n+1} \end{array} \right) - \left(\begin{array}{c} \boldsymbol{0} \\ y_{n+1} \end{array} \right) \right\|.$$

   Use (2) to obtain

$$\left\| \left(\begin{array}{c} \boldsymbol{x}_0 \\ 0 \end{array} \right) - \left(\begin{array}{c} \boldsymbol{y}_0 \\ 0 \end{array} \right) \right\| = \|\boldsymbol{x}_0 - \boldsymbol{y}_0\| < \rho_0,$$

   and

$$\left\| \left(\begin{array}{c} \boldsymbol{0} \\ x_{n+1} \end{array} \right) - \left(\begin{array}{c} \boldsymbol{0} \\ y_{n+1} \end{array} \right) \right\| \leqslant \left\| \left(\begin{array}{c} 0 \\ \boldsymbol{x}_1 \end{array} \right) - \left(\begin{array}{c} 0 \\ \boldsymbol{y}_1 \end{array} \right) \right\| = \|\boldsymbol{x}_1 - \boldsymbol{y}_1\| < \rho_1.$$

   Conclude that for $\|\boldsymbol{x}_0 - \boldsymbol{y}_0\| < \rho_0$ and $\|\boldsymbol{x}_1 - \boldsymbol{y}_1\| < \rho_1$, the norm of the higher dimensional difference vector satisfies

$$\|\boldsymbol{x} - \boldsymbol{y}\| \leqslant \rho_0 + \rho_1,$$

in which case the difference in the $(n+1)$-order finite difference satisfies

$$|f[x_0,...,x_{n+1}]-f[y_0,...,y_{n+1}]|=\frac{1}{(n+1)h}|(f[x_1,...,x_{n+1}]-f[y_1,...y_{n+1}])-(f[x_0,...,x_n]-f[y_0,...,y_n])|\leqslant$$

$$\leqslant\frac{|f[x_1,...,x_{n+1}]-f[y_1,...y_{n+1}]|}{(n+1)h}+\frac{|f[x_0,...,x_n]-f[y_0,...,y_n]|}{(n+1)h}\leqslant\frac{1}{(n+1)h}\left[\frac{(n+1)\,h\sigma}{2}+\frac{(n+1)\,h\sigma}{2}\right]=\sigma,$$

hence, indeed $\mathcal{P}(n+1)$ is true, and by complete induction: $f\in C(\mathbb{R})\Rightarrow f[x_0,...,x_n]\in C(\mathbb{R}^{n+1})$. □

Note: We've assumed equally spaced nodes, $x_j=x_0+jh$. Go through the proof and convince yourself that the arguments still hold when the nodes are distinct, but not necessarily equally spaced.

1.b. K&C p.336, 6.2.3 (convergence characterization)

Solution: The solution to this problem brings out the dangers of imprecise formulation, in this case of the convergence process.

1. *State the problem in mathematical terms, specifying the notation.* If $f\in C^n[a,b]$

$$x_0\in(a,b)\text{ and }x_1,...,x_n\to x_0\Rightarrow f[x_0,...,x_n]\to\frac{f^{(n)}(x_0)}{n!} \tag{3}$$

2. *Apply definitions.*
   i. Continuity $f^{(k)}$. $\forall\varepsilon>0$, $\exists\delta$ such that for $\forall x,y\in(a,b)$ satisfying $|x-y|<\delta$, the inequality $\left|f^{(k)}(x)-f^{(k)}(y)\right|<\varepsilon$ is true for differentiation orders $k=0,1,...,n$.
   ii. Convergence. The nature of the convergence process $x_1,...,x_n\to x_0$ is not stated in the problem. We'll show how two different interpretations can lead to different results.
      a) Consider a positive valued sequence $\{a_m>0\}_{m\in\mathbb{N}}$ that converges to zero, i.e., $\forall\varepsilon>0$, $\exists M_\varepsilon$ such that for $m>M_\varepsilon$ $|a_m|<\varepsilon$. The sequences $\{x_{k,m}=x_0+ka_m\}_{m\in\mathbb{N}}$ also converge to zero, and $x_{n,m}>x_{n-1,m}>...>x_{1,m}>x_0$, mainting distinct points. In this case the points $x_1,...,x_n$ converge to x_0 maintaining a uniform distance between x_j and x_{j-1}, such that

$$x_{n+1,m}-x_0=(n+1)a_m=(n+1)(x_{1,m}-x_0)$$

      b) Now consider the sequences $\{x_{k,m}=x_0+k^2a_m\}_{m\in\mathbb{N}}$ that again converge to zero as $m\to\infty$, with $x_{n,m}>x_{n-1,m}>...>x_{1,m}>x_0$, but with nonuniform distances between the points, such that for a given m

$$x_{n+1,m}-x_0=(n+1)^2a_m=(n+1)^2(x_{1,m}-x_0).$$

   iii. Differentiability. Using a formulation based on sequences, f is differentiable at x_0 if for $\forall\{a_m\}_{m\in\mathbb{N}}$ that converges to zero, the sequence

$$\frac{f(x_0+a_m)-f(x_0)}{a_m}$$

   converges.
   iv. Divided differences. $f[x_j]=f(x_j)$,

$$f[x_j,...,x_{j+l+1}]=\frac{f[x_{j+1},...,x_{j+l+1}]-f[x_j,...,x_{j+l}]}{x_{j+l+1}-x_j}.$$

3. *Identify a proof strategy.* Complete induction, denote (3) as $\mathcal{P}(n)$.
4. *Apply your proof strategy.* $\mathcal{P}(1)$: $f[x_0,x_1]=(f(x_1)+f(x_0))/(x_1-x_0)=(f(x_1+a_m)+f(x_0))/a_m\to f'(x_0)$

   Assume $\mathcal{P}(n)$ true, which implies $f[x_0,...,x_n]\to\frac{f^{(n)}(x_0)}{n!}$ and $f[x_1,...,x_{n+1}]\to\frac{f^{(n)}(x_1)}{n!}$

   Establish $\mathcal{P}(n+1)$. For the choice of sequences $\{x_{k,m}=x_0+ka_m\}_{m\in\mathbb{N}}$ and using divided difference, convergence definitions

$$f[x_0,...,x_{n+1}]=\frac{f[x_1,...,x_{n+1}]-f[x_0,...,x_n]}{x_{n+1}-x_0}=\frac{f[x_1,...,x_{n+1}]-f[x_0,...,x_n]}{(n+1)a_m}$$

Using $\mathcal{P}(n)$

$$\lim_{m\to\infty}\frac{f[x_1,...,x_{n+1}]-f[x_0,...,x_n]}{(n+1)a_m}=\lim_{m\to\infty}\frac{f^{(n)}(x_0+a_m)-f^{(n)}(x_0)}{(n+1)n!a_m}=\frac{f^{(n+1)}(x_0)}{(n+1)!},$$

thus proving $\mathcal{P}(n+1)$. However, if we use the sequences $\{x_{k,m}=x_0+k^2\,a_m\}_{m\in\mathbb{N}}$, repeating the above calculations gives

$$\lim_{x_1,...,x_n\to x_0}f[x_0,...,x_{n+1}]=\lim_{m\to\infty}\frac{f^{(n)}(x_0+a_m)-f^{(n)}(x_0)}{(n+1)^2a_m}=\frac{1}{n+1}\cdot\frac{f^{(n+1)}(x_0)}{(n+1)!},$$

contradicting the general statement! Take this as an example of the necessity of precise mathematical language.

The remaining problems are solved in a concise manner, without the extensive pedagogical explanations provided above, and in the style that is appropriate for homework or answering an examination question.

1.c. K&C p.336, 6.2.5. (identification of identical polynomials) $p\in P_n\Rightarrow p(x)=\sum_{i=0}^n p[x_0,x_1,...,x_i]\prod_{j=0}^{i-1}(x-x_j)$.

Solution. The Newton polynomial interpolant of data $\mathcal{D}=\{(x_i,\ y_i=p(x_i)),\ i=0,\ ...,\ n\}$ is $q(x)=\sum_{i=0}^{n} p[x_0,\ x_1,\ ...,\ x_i]\prod_{j=0}^{i-1}(x-x_j)$, a polynomial of degree n. Polynomials of degree n with the same coefficient $p[x_0,x_1,...,x_i]$ for x^n, that pass through the same $n+1$ points in $\mathcal{D}$ are identical, since otherwise $r(x)=p(x)-q(x)$ a polynomial of degree n would have $n+1$ roots (contradiction).

1.d. K&C p.336, 6.2.6. (calculations with divided differences) $\mathcal{P}(n)$: $(\alpha f+\beta g)[x_0,...,.x_n]=\alpha f[x_0,...,.x_n]+\beta g[x_0,...,.x_n]$.

Solution. By complete induction. $\mathcal{P}(0)$: $(\alpha f+\beta g)[x_0]=(\alpha f+\beta g)(x_0)=\alpha f(x_0)+\beta g(x_0)$. Assume $\mathcal{P}(n)$. Consider $\mathcal{P}(n+1)$ and apply definition of divided differences

$$(\alpha f+\beta g)[x_0,...,.x_{n+1}]=\frac{(\alpha f+\beta g)[x_1,...,.x_{n+1}]-(\alpha f+\beta g)[x_0,...,.x_n]}{x_{n+1}-x_0}. \tag{4}$$

By $\mathcal{P}(n)$:

$$(\alpha f+\beta g)[x_1,...,.x_{n+1}]=\alpha f[x_1,...,.x_{n+1}]+\beta g[x_1,...,.x_{n+1}] \tag{5}$$

$$(\alpha f+\beta g)[x_0,...,.x_n]=\alpha f[x_0,...,.x_n]+\beta g[x_0,...,.x_n] \tag{6}$$

Combine (4-6) to obtain

$$(\alpha f+\beta g)[x_0,...,.x_{n+1}]=\alpha\frac{f[x_1,...,.x_{n+1}]-f[x_0,...,.x_n]}{x_{n+1}-x_0}+\beta\frac{g[x_1,...,.x_{n+1}]-g[x_0,...,.x_n]}{x_{n+1}-x_0},$$

establishing $\mathcal{P}(n+1)\,\square$.

2.a K&C, p.336, 6.2.7. (analogy derivative-divided difference) Solution. The property analogous to $(fg)'=f'\,g+fg'$ for divided differences would be

$$(fg)[x_0,x_1]=f[x_0,x_1]\,g(\alpha)+f(\beta)g[x_0,x_1] \tag{7}$$

with the evaluation points α,β as yet unknown. By definition of divided differences

$$E_1=(fg)[x_0,x_1]=\frac{f(x_1)\,g(x_1)-f(x_0)g(x_0)}{x_1-x_0},$$

$$E_2=f[x_0,x_1]\,g(\alpha)+f(\beta)g[x_0,x_1]=\frac{f(x_1)-f(x_0)}{x_1-x_0}g(\alpha)+f(\beta)\frac{g(x_1)-g(x_0)}{x_1-x_0}$$

Set $E_1=E_2$ to see if α,β can be found s.t. (7) true for $\forall f,g$:

$$f(x_1)\,g(x_1)-f(x_0)g(x_0)=[f(x_1)-f(x_0)]g(\alpha)+f(\beta)[g(x_1)-g(x_0)]. \tag{8}$$

Choose $\alpha=x_0$ and $\beta=x_1$ to obtain

$$[f(x_1)-f(x_0)]g(x_0)+f(x_1)[g(x_1)-g(x_0)]=f(x_1)g(x_1)-f(x_0)g(x_0),$$

hence $E_1=E_2$. Choosing $\alpha=x_1$ and $\beta=x_0$ also works

$$[f(x_1)-f(x_0)]g(x_1)+f(x_0)[g(x_1)-g(x_0)]=f(x_1)g(x_1)-f(x_0)g(x_0).$$

2.b. K&C, p.336 6.2.13. (Calculation with sums, divided differences) $\mathcal{P}(n)$: $(fg)[x_0,...,x_n]=\sum_{k=0}^{n} f[x_0,...,x_k]g[x_k,...,x_n]$.

Solution. $\mathcal{P}(1)$: $(fg)[x_0,x_1]=f[x_0]g[x_0,x_1]+f[x_0,x_1]g[x_1]=f(x_0)g[x_0,x_1]+f[x_0,x_1]g(x_1)$, established in 2.a.

Assume $\mathcal{P}(n)$. Set $H=x_{n+1}-x_0$, $h_k=x_k-x_{k-1}$, and use divided difference definition and $\mathcal{P}(n)$ to compute

$$\begin{aligned}
H\cdot(fg)[x_0...x_{n+1}] &= (fg)[x_1...x_{n+1}]-(fg)[x_0...x_n]=\\
=\sum_{k=1}^{n+1} f[x_1...x_k]g[x_k...x_{n+1}] &- \sum_{k=0}^{n} f[x_0...x_k]g[x_k...x_n]=\\
=\sum_{k=0}^{n} f[x_1...x_{k+1}]g[x_{k+1}...x_{n+1}] &- \sum_{k=0}^{n} f[x_0...x_k]g[x_k...x_n]=\\
=\sum_{k=0}^{n}(f[x_0...x_k]+(x_{k+1}-x_0)f[x_0...x_{k+1}])g[x_{k+1}...x_{n+1}] &- \sum_{k=0}^{n} f[x_0...x_k]g[x_k...x_n]=\\
=\sum_{k=0}^{n} f[x_0...x_k]g[x_{k+1}...x_{n+1}]-\sum_{k=0}^{n} f[x_0...x_k]g[x_k...x_n] &+ \sum_{k=0}^{n}(x_{k+1}-x_0)f[x_0...x_{k+1}]g[x_{k+1}...x_{n+1}]=\\
=\sum_{k=0}^{n} f[x_0...x_k](g[x_{k+1}...x_{n+1}]-g[x_k...x_n]) &+ \sum_{k=0}^{n}(x_{k+1}-x_0)f[x_0...x_{k+1}]g[x_{k+1}...x_{n+1}]=\\
=\sum_{k=0}^{n}(x_{n+1}-x_k)f[x_0...x_k]g[x_k...x_{n+1}] &+ \sum_{k=1}^{n+1}(x_k-x_0)f[x_0...x_k]g[x_k...x_{n+1}]=\\
(x_{n+1}-x_0)f[x_0]g[x_0...x_{n+1}]+\sum_{k=0}^{n}(x_{n+1}-x_k+x_k-x_0) &\cdot f[x_0...x_k]g[x_k...x_{n+1}]+(x_{n+1}-x_0)f[x_0...x_{n+1}]g(x_{n+1})\\
&= H\sum_{k=0}^{n+1} f[x_0...x_k]g[x_k...x_{n+1}]\,\square
\end{aligned}$$

3.a. K&C, p.336, 6.2.8. Direct result of uniqueness of global polynomial interpolant.

3.b. K&C, p.336, 6.2.9. Equate coefficients of x^n in Lagrange and Newton forms of the polynomial interpolant.

3.c. K&C, p.336, 6.2.14. From $\alpha_i^{-1} = (x_i - x_0)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_n)$ deduce $\text{sign}(\alpha_i^{-1}) = (1)^i(-1)^{n-i}$. From $\alpha_{i+1}^{-1} = (x_{i+1} - x_0)...(x_{i+1} - x_i)(x_{i+1} - x_{i+2})...(x_{i+1} - x_n)$ deduce $\text{sign}(\alpha_{i+1}^{-1}) = (1)^{i+1}(-1)^{n-i-1}$. Compute

$$\text{sign}(\alpha_i^{-1}\alpha_{i+1}^{-1}) = (-1)^{2(n-1)-1} = -1,$$

hence alternating signs.

3.d. K&C, p.337, 6.15. In $f[x_0, ..., x_{n+1}] = \sum_{i=0}^n \alpha_i f(x_i)$ choose $f(x) = x^n$ to obtain $f[x_0, ..., x_{n+1}] = \sum_{i=0}^n \alpha_i x_i^n$. Prove $\mathcal{P}(n)$: $f_n(x) = x^n \Rightarrow f_n[x_0, ..., x_{n+1}] = 1$, by complete induction. $\mathcal{P}(1)$: $f_1(x) = x \Rightarrow f_1[x_0, x_1] = \frac{x_1 - x_0}{x_1 - x_0} = 1$. Assume $\mathcal{P}(n)$ true. Consider $f_{n+1}(x) = x^{n+1} = f_n(x)\, g(x)$, with $g(x) = x$ such that divided differences of order 2 and higher are null, and apply result from Problem 2.b

$$(f_n g)[x_0, ..., x_{n+1}] = \sum_{k=0}^{n+1} f_n[x_0, ..., x_k] g[x_k, ..., x_{n+1}] = f_n[x_0, ..., x_n] g[x_n, x_{n+1}].$$

By $\mathcal{P}(n)$, $f_n[x_0, ..., x_n] = 1$, and $g[x_n, x_{n+1}] = 1$, establishing $\mathcal{P}(n+1)$.

In $f[x_0, ..., x_{n+1}] = \sum_{i=0}^n \alpha_i f(x_i)$ choose $f(x) = 1$ to obtain $\sum_{i=0}^n \alpha_i = 1$ for $n = 0$, $\sum_{i=0}^n \alpha_i = 1$ for $n > 0$.

4.a. K&C, p.338, 6.2.27. In $f[x_0, ..., x_m] = \sum_{j=0}^m \alpha_j f(x_j)$ choose $x_j = j$ such that

$$\alpha_j = \prod_{\substack{k=0\\k\neq j}}^m (x_j - x_k)^{-1} = \prod_{\substack{k=0\\k\neq j}}^m (j-k)^{-1} = \frac{1}{j\cdot(j-1)\cdot...\cdot 1\cdot(-1)\cdot...(j-m)} = \frac{(-1)^{m-j}}{j!\,(m-j)!} = \frac{(-1)^{m-j} m(m-1)...(m-j+1)}{j!m!}.$$

Since

$$\binom{m}{j} = \frac{m(m-1)...(m-j+1)}{j!},$$

it results that

$$m!\, f[0, 1, ..., m] = \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} f(x_j).$$

4.b. K&C, p.338, 6.2.28. The definition of a divided difference by Hermite-Genocchi is

$$f[x_0, ..., x_n] = \int_{S_n} f^{(n)}(u_0 x_0 + ... + u_n x_n)\, \mathrm{d}\boldsymbol{u} \tag{9}$$

with $\mathrm{d}\boldsymbol{u} = \mathrm{d}u_n \cdot ... \cdot \mathrm{d}u_0$, and the simplex S_n defined by

$$S_n = \left\{ \boldsymbol{u} = \begin{pmatrix} u_0 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^{n+1} : u_i \geqslant 0, \sum_{i=0}^n u_i = 1 \right\}.$$

Note that the simplex is an n-dimensional vector space embedded in $\mathbb{R}^{n+1}$.

For example the first-order divided difference is computed as

$$f[x_0, x_1] = \int_{S_1} f'(u_0 x_0 + u_1 x_1) \mathrm{d}u_1 \mathrm{d}u_0.$$

The integral is over the line $\{(1 - u_1, u_1) : 0 \leqslant u_1 \leqslant 1\}$, and replacing $u_0 = 1 - u_1$ leads to

$$f[x_0, x_1] = \int_0^1 f'(x_0 + u_1(x_1 - x_0))\, \mathrm{d}u_1 = \left[\frac{f(x_0 + u_1(x_1 - x_0))}{x_1 - x_0} \right]_{u_1=0}^{u_1=1} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

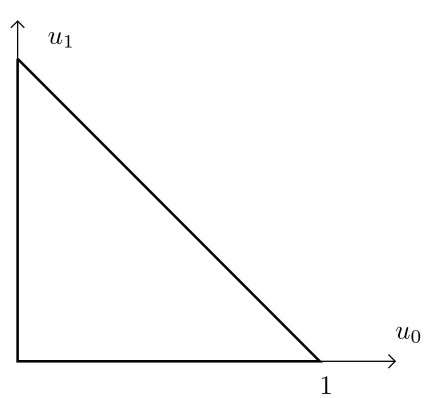

Figure 1. Simplicial integration domain for $n = 1$.

The recursive formula that is to be proved is

$$f[x_0, ..., x_{n+1}] = \frac{f[x_1, ..., x_{n+1}] - f[x_0, ..., x_n]}{x_{n+1} - x_0}. \tag{10}$$

Apply (9) to divided differences on rhs to obtain

$$\frac{f[x_1, ..., x_{n+1}] - f[x_0, ..., x_n]}{x_{n+1} - x_0} = \frac{1}{x_{n+1} - x_0}\left[\int_{S_n} f^{(n)}(u_0 x_1 + ... + u_n x_{n+1})\, \mathrm{d}\boldsymbol{u} - \int_{S_n} f^{(n)}(u_0 x_0 + ... + u_n x_n)\, \mathrm{d}\boldsymbol{u}\right].$$

Apply (9) to lhs of (10) to obtain

$$f[x_0, ..., x_{n+1}] = \int_{S_{n+1}} f^{(n+1)}(u_0 x_1 + ... + u_n x_{n+1})\, \mathrm{d}u_{n+1}\, \mathrm{d}\boldsymbol{u}$$

with the integration domain S_{n+1}, an $n+1$ dimensional simplex embedded in $\mathbb{R}^{n+2}$. Replace $u_0 = 1 - u_1 - ... - u_{n+1}$, to obtain

$$f[x_0, ..., x_{n+1}] \;=\; \int_0^1 \int_0^{1-u_1} \int_0^{1-u_1-u_2} ... \int_0^{1-u_1-...-u_n} f^{(n+1)}(x_0 + ... + u_{n+1}(x_{n+1} - x_0))\, \mathrm{d}u_{n+1} ... \mathrm{d}u_1. \tag{11}$$

In (11) carry out the integration w.r.t. u_{n+1} to obtain

$$\begin{aligned}
f[x_0, ..., x_{n+1}] &= \frac{1}{x_{n+1} - x_0}\int_0^1 \int_0^{1-u_1} ... \int_0^{1-\sum_{i=1}^{n-1} u_i} \left[f^{(n)}(x_0 + ... + u_n (x_n - x_0) + u_{n+1}(x_{n+1} - x_0))\right]_{u_{n+1}=0}^{u_{n+1}=1-\sum_{i=1}^{n} u_i} \mathrm{d}u_n ... \mathrm{d}u_1\\
&= \frac{1}{x_{n+1} - x_0}\int_0^1 \int_0^{1-u_1} ... \int_0^{1-\sum_{i=1}^{n-1} u_i} \left[f^{(n)}\left(x_0 + ... + u_n (x_n - x_0) + \left(1 - \sum_{i=1}^{n} u_i\right) \cdot (x_{n+1} - x_0)\right)\right] \mathrm{d}u_n ... \mathrm{d}u_1 -\\
&\quad \frac{1}{x_{n+1} - x_0}\int_0^1 \int_0^{1-u_1} ... \int_0^{1-\sum_{i=1}^{n-1} u_i} \left[f^{(n)}(x_0 + ... + u_n (x_n - x_0) + (0) \cdot (x_{n+1} - x_0))\right] \mathrm{d}u_n ... \mathrm{d}u_1\\
&= \frac{1}{x_{n+1} - x_0}\left[\int_{S_n} f^{(n)}(u_0 x_1 + ... + u_n x_{n+1})\, \mathrm{d}\boldsymbol{u} - \int_{S_n} f^{(n)}(u_0 x_0 + ... + u_n x_n)\, \mathrm{d}\boldsymbol{u}\right]\square
\end{aligned}$$

5. (Computational problem) Construct the generalization of Newton interpolation $p_m(x, y)$ of bivariate data $\mathcal{D} = \{(x_{ij}, y_{ij}, z_{ij}), i, j = 0, 1, ..., n\}$, implement an efficient program to determine and evaluate the interpolant. Test on the functions f, g: $[-1, 1] \times [-1, 1] \to \mathbb{R}$.

$$f(x, y) = \frac{1}{1 + 25(x^2 + y^2)}, g(x, y) = \exp[-(x^2 + y^2)].$$

Assume that $(x_{ij}, y_{ij}) \in \{x_0, ..., x_n\} \times \{y_0, ..., y_n\}$.

Solution. Recall that for univariate polynomial interpolation of data $\mathcal{D}_1 = \{(x_i, y_i), i = 0, 1, ..., n\}$, evaluation of the monomial basis set $\{1, x, ..., x^n\}$ at the nodes x_i led to the Vandermonde matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & x_0 & ... & x_0^n\\ 1 & x_1 & ... & x_1^n\\ \vdots & \vdots & \ddots & \vdots\\ 1 & x_n & ... & x_n^n \end{pmatrix},$$

and the coefficients $\boldsymbol{c} \in \mathbb{R}^{n+1}$ of the interpolating polynomial are the solution of the system

$$\boldsymbol{A}\boldsymbol{c} = \boldsymbol{y}.$$

The matrix is full with this choice of basis set and the computational cost is $\mathcal{O}(n^3)$. Choosing the Newton triangular basis set $\{1, x - x_0, (x - x_0)(x - x_1), ..., (x - x_0) \cdot ... \cdot (x - x_{n-1})\}$ leads to the triangular matrix

$$\boldsymbol{N} = \begin{pmatrix} 1 & 0 & 0 & ... & 0\\ 1 & x_1 - x_0 & 0 & ... & 0\\ 1 & x_2 - x_0 & (x_2 - x_0)(x_2 - x_1) & ... & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_n - x_0 & (x_n - x_0)(x_n - x_1) & ... & \prod_{i=0}^{n-1} (x_n - x_i) \end{pmatrix},$$

and solving the interpolation condition system

$$\boldsymbol{N}\boldsymbol{d} = \boldsymbol{y},$$

by forward substitution only costs $\mathcal{O}(n^2)$ operations.

A similar approach for bivariate interpolation requires definition of an analogous set of triangular basis functions, with $(n+1)^2$ members since there are $(n+1)^2$ interpolation conditions. Consider the set of functions

$$S = \left\{ \begin{array}{lllll} 1 & x-x_0 & (x-x_0)(x-x_1) & \dots & \prod_{i=0}^{n-1}(x-x_i) \\ y-y_0 & (x-x_0)(y-y_0) & (x-x_0)(x-x_1)(y-y_0) & \dots & \prod_{i=0}^{n-1}(x-x_i)(y-y_0) \\ (y-y_0)(y-y_1) & (x-x_0)(y-y_0)(y-y_1) & (x-x_0)(x-x_1)(y-y_0)(y-y_1) & \dots & \prod_{i=0}^{n-1}(x-x_i)(y-y_0)(y-y_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \prod_{i=0}^{n-1}(y-y_i) & (x-x_0)\prod_{i=0}^{n-1}(y-y_i) & (x-x_0)(x-x_1)\prod_{i=0}^{n-1}(y-y_i) & \dots & \prod_{i=0}^{n-1}(x-x_i)\prod_{i=0}^{n-1}(y-y_i) \end{array} \right\}.$$

The set S can be presented in linear form as

$$\left\{ 1 \quad x-x_0 \quad \dots \quad \prod_{i=0}^{n-1}(x-x_i) \quad y-y_0 \quad (x-x_0)(y-y_0) \quad \dots \quad \prod_{i=0}^{n-1}(x-x_i)(y-y_0) \quad \dots \quad \prod_{i=0}^{n-1}(y-y_i) \quad \dots \quad \prod_{i=0}^{n-1}(x-x_i)\prod_{i=0}^{n-1}(y-y_i) \right\}$$

Evaluation of the basis set at the nodes $(x_{ij}, y_{ij}) \in \{x_0, \dots, x_n\} \times \{y_0, \dots, y_n\} = \{(x_0, y_0), (x_1, y_0), \dots, (x_n, y_0), \dots, (x_n, y_n)\}$ leads to the matrix

$$\boldsymbol{B} = \left(\begin{array}{ccccc} \boldsymbol{N} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{N} & \boldsymbol{N}(y_1-y_0) & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{N} & \boldsymbol{N}(y_2-y_0) & \boldsymbol{N}(y_2-y_0)(y_2-y_1) & \dots & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{N} & \boldsymbol{N}(y_n-y_0) & \boldsymbol{N}(y_n-y_0)(y_n-y_1) & \dots & \boldsymbol{N}\prod_{i=0}^{n-1}(y-y_i) \end{array} \right)$$

with $\boldsymbol{N}$ the univariate Newton basis matrix, and $\boldsymbol{0} \in \mathbb{R}^{(n+1)\times(n+1)}$ the null matrix. A block triangular structure is obtained, and imposing the interpolation conditions leads to

$$\boldsymbol{B}\boldsymbol{c} = \boldsymbol{z}.$$

Implementation.

```
octave> function z=f(x,y)
          z=1./(1.+25.*(x.^2+y.^2));
        endfunction;
octave> n=4; xl=yl=linspace(-1,1,n+1)'; o=ones(size(xl));
octave> [x,y]=meshgrid(xl,yl);
octave> z=f(x,y);
octave> B=zeros((n+1)^2);
octave> N=zeros(n+1); N(:,1)=1;
octave> for j=2:n+1
          dx = xl - xl(j-1)*o;
          N(:,j) = dx.*N(:,j-1);
        end;
octave> format rat; disp(N);
           1           0          -0           0          -0
           1         1/2           0          -0           0
           1           1         1/2           0          -0
           1         3/2         3/2         3/4           0
           1           2           3           3         3/2

octave>
octave> for i=1:n+1:(n+1)^2
          B(i:i+n,1:n+1) = N;
        end;
octave> cd /home/student/courses/MATH661/homework; spy(flip(B)); print -mono -deps spy1stblk.eps;
octave>
```

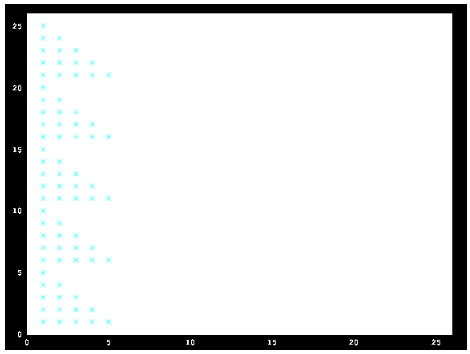

Figure 2. spy(B) after filing first column block

```
octave> for j=2:n+1
          dy = yl - yl(j-1)*o;
          for i=j:n+1
            B( (i-1)*(n+1)+1:i*(n+1) , (j-1)*(n+1)+1:j*(n+1) ) = dy(i)*N;
          end;
        end;
octave> spy(flip(B)); print -mono -deps spyB.eps;
octave>
```

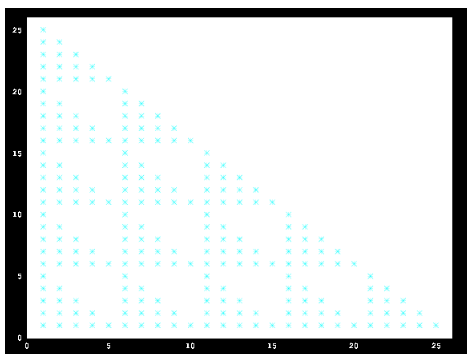

Figure 3. spy(B)

```
octave> rank(B)
```

25

```
octave> cond(B)
```

1467.464

```
octave> c = B \ reshape(z,(n+1)^2,1);
octave> function z=newt2(c,x,y,xl,yl)
          n = max(size(xl))-1;
          S = ones(n+1,n+1);
          for i=2:n+1
            S(i,1) = S(i-1,1)*(y-yl(i-1));
          end;
          for j=2:n+1
            S(:,j) = S(:,j-1)*(x-xl(j-1));
          end;
          z = reshape(S,(n+1)^2,1)'*c;
        endfunction;
octave> newt2(c,xl(1),yl(2),xl,yl)-z(1,2)
```

0

```
octave> newt2(c,xl(3),yl(4),xl,yl)-z(3,4)
```

$2.7756e-16$

```
octave> nf=100; xfl=yfl=linspace(-1,1,nf)'; [xf,yf]=meshgrid(xfl,yfl); zf=f(xf,yf);
octave> mesh(xfl,yfl,zf);  print -deps runge.eps;
octave> zi=zeros(nf,nf);
```

```
octave> for i=1:nf
          for j=1:nf
            zi(i,j)=newt2(c,xfl(i),yfl(j),xl,yl);
          end
        end;
octave> figure(2); mesh(xfl,yfl,zi);  print -deps rungei.eps;
octave>
```

Results are rendered in Fig. 4

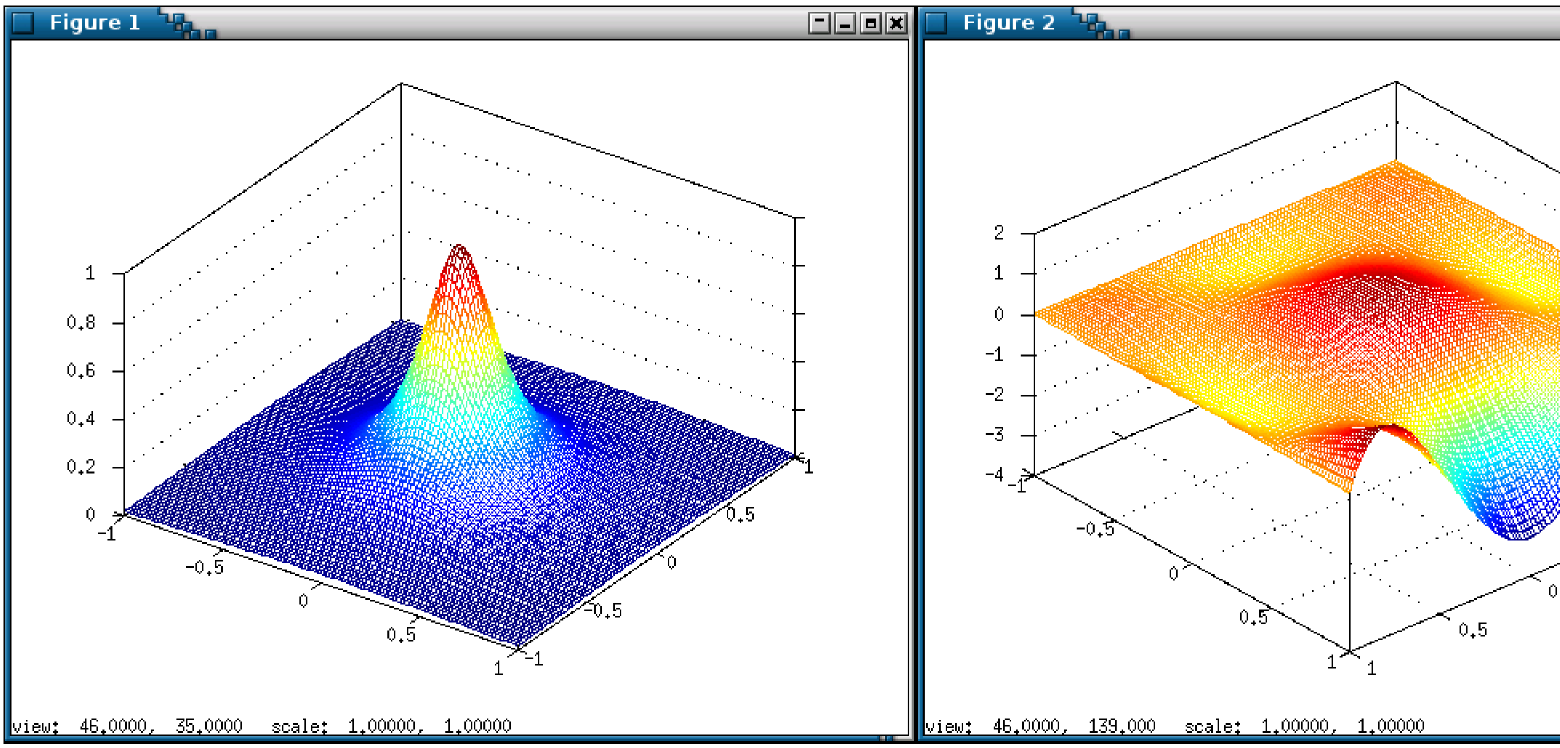

Figure 4. Runge function (left), Bivariate, uniform spaced polynomial interpolation results (right)

Repeating the calculations for $g(x, y) = \exp[-(x^2 + y^2)]$

```
octave> function z=g(x,y)
          z=exp(-(x.^2+y.^2));
        endfunction;
octave> n=4; xl=yl=linspace(-1,1,n+1)'; o=ones(size(xl));
octave> [x,y]=meshgrid(xl,yl);
octave> z=f(x,y);
octave> B=zeros((n+1)^2);
octave> N=zeros(n+1); N(:,1)=1;
octave> for j=2:n+1
          dx = xl - xl(j-1)*o;
          N(:,j) = dx.*N(:,j-1);
        end;
octave> for i=1:n+1:(n+1)^2
          B(i:i+n,1:n+1) = N;
        end;
octave> for j=2:n+1
          dy = yl - yl(j-1)*o;
          for i=j:n+1
            B( (i-1)*(n+1)+1:i*(n+1) , (j-1)*(n+1)+1:j*(n+1) ) = dy(i)*N;
          end;
        end;
octave> c = B \ reshape(z,(n+1)^2,1);
```

```
octave> newt2(c,xl(3),yl(2),xl,yl)-z(3,2)
   0
octave> newt2(c,xl(2),yl(1),xl,yl)-z(2,1)
   0
octave> nf=100; xfl=yfl=linspace(-1,1,nf)'; [xf,yf]=meshgrid(xfl,yfl); zf=f(xf,yf);
octave> mesh(xfl,yfl,zf);  print -deps exp2d.eps;
octave> zi=zeros(nf,nf);
octave> for i=1:nf
          for j=1:nf
            zi(i,j)=newt2(c,xfl(i),yfl(j),xl,yl);
          end
        end;
octave> figure(3); mesh(xfl,yfl,zi);  print -deps exp2di.eps;
octave>
```