MATH661: Homework 3
Issued: 9/30/15
Due: 10/7/15

Notation: K&C = Kincaid & Cheney textbook

1.a. K&C p.488, 7.2.4: Verify for $f(x) \in \{1, x, x^2, x^3, x^4\}$

$$\int_0^1 x^k \, \mathrm{d}x = \frac{1}{k+1}$$

$$\int_0^1 \mathrm{d}x = 1 = \frac{1}{90}[7 + 32 + 12 + 32 + 7] \checkmark$$

Use Octave for arithmetic for $k = 1, 2, 3, 4$

```
octave> function err=quad(k)
          err=1/(k+1)-(32/4**k+12/2**k+32*(3/4)**k+7)/90;
        end;
octave> disp([quad(1) quad(2) quad(3) quad(4)]);
   0   0   0   0
octave>
```

1.b. K&C p.488, 7.2.5: Use transformation $x = (b-a)t + a \Rightarrow \mathrm{d}x = (b-a)\mathrm{d}t$ to obtain

$$\int_a^b f(x)\, \mathrm{d}x = (b-a)\int_0^1 f((b-a)t + a)\, \mathrm{d}t = \frac{1}{90}\left[7 f(a) + 32 f\left(\frac{3a+b}{4}\right) + 12 f\left(\frac{a+b}{2}\right) + 32 f\left(\frac{a+3b}{4}\right) + 7 f(b)\right]$$

1.c. K&C p.489, 7.2.6:

$$\int_0^1 \frac{\mathrm{d}t}{t+1} = [\ln(t+1)]_{t=0}^{t=1} = \ln 2 \cong \frac{1}{90}\left[7 + 32\frac{1}{1+\frac{1}{4}} + 12\frac{1}{1+\frac{1}{2}} + 32\frac{1}{1+\frac{3}{4}} + 7\frac{1}{1+1}\right]$$

```
octave> log(2.)
  0.69315
octave> (7+32*4/5+12*2/3+32*4/7+7/2)/90.
  0.69317
octave> relerr=abs((7+32*4/5+12*2/3+32*4/7+7/2)/90.-log(2.))/log(2.); disp(relerr);
   3.9562e-05
octave>
```

2.a. K&C, p.489, 7.2.8: Evaluate integral to obtain

$$\int_0^1 [a e^x + b\cos(\pi x/2)]\, \mathrm{d}x = (e-1)a + \frac{2}{\pi}b = A_1(a+b) + A_2(a e) = (A_1 + e A_2)a + A_1 b$$

Since a, b are arbitrary the above equality is satisfied when

$$A_1 + e A_2 = e - 1, \frac{2}{\pi} = A_1 \Rightarrow A_2 = \frac{e - 1 - 2/\pi}{e}.$$

2.b. K&C, p.489, 7.2.9: Evaluate integral to obtain

$$\int_0^{2\pi} (a + b\cos x)\, \mathrm{d}x = 2\pi a = A_1(a+b) + A_2(a-b) = 2A_1 a + (A_1 - A_2)b.$$

Since a, b are arbitrary the above equality is satisfied when

$$A_1 = \pi, A_2 = \pi.$$

For $f(x) = \sum_{k=0}^n [a_k \cos(2k+1)x + b_k \sin kx]$

$$\int_0^{2\pi} f(x)\, \mathrm{d}x = 0$$

due to alternating half-periods of opposite signed areas. Evaluation of f at quadrature nodes gives

$$f(0)=\sum_{k=0}^{n} a_k,\, f(\pi)=-\sum_{k=0}^{n} a_k$$

hence

$$A_1 f(0)+A_2 f(\pi)=0$$

the exact integral value.

3.a. K&C, p.491 7.2.26: Chebyshev polynomials of second kind were introduced as

$$U_{n+1}(x)=\frac{\sin[(n+2)\theta]}{\sin\theta},$$

with $x=\cos\theta$. Verify start of recursion using above definition

$$U_0(x)=\frac{\sin[(-1+2)\theta]}{\sin\theta}=1\checkmark$$

$$U_1(x)=\frac{\sin[(0+2)\theta]}{\sin\theta}=\frac{2\sin\theta\cos\theta}{\sin\theta}=2\cos\theta\checkmark$$

Verify recursion

$$\frac{\sin[(n+2)\theta]}{\sin\theta}=2\cos\theta\frac{\sin[(n+1)\theta]}{\sin\theta}-\frac{\sin(n\theta)}{\sin\theta}\Rightarrow\sin[(n+2)\theta]+\sin(n\theta)=2\sin[(n+1)\theta]\cos\theta\checkmark$$

a well-known trigonometric identity.

3.b. K&C, p.491, 7.2.27: Use transformation $\cos\theta=x\Rightarrow-\sin\theta\,\mathrm{d}\theta=\mathrm{d}x$

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\,\mathrm{d}x=\int_0^{\pi}\frac{\sin[(n+1)\theta]}{\sin\theta}\frac{\sin[(m+1)\theta]}{\sin\theta}\sin^2\theta\,\mathrm{d}\theta=\frac{1}{2}\int_0^{\pi}[\cos[(n-m)\theta]-\cos[(m+n+2)\theta]]\,\mathrm{d}\theta=\frac{\pi}{2}\delta_{mn}.\checkmark$$

3.c. K&C, p.491, 7.2.28

4.a. K&C, p.498, 7.3.3
4.b. K&C, p.498, 7.3.8
4.c. K&C, p.498, 7.3.22

5. (Computational problem) K&C, p. 507, 7.4.1.