MATH661 Homework 2 - Multivariate approximation

Posted: Sep 6 Due: 11:55PM, Sep 20

1 Problem statement

Many physical phenomena are determined by the shape of surfaces separating one domain from another. Examples include:

- flow around a body;
- inclusions of one phase in another, e.g., air bubble in water;
- accumulation of defects in a crystal at boundaries, e.g., formation of metal grains.

A first step in modeling such phenomena is to approximate the bounding surface in an accurate and computationally efficient manner. The parametric form of the surface is $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, with components $(f_1(u,v), f_2(u,v), f_3(u,v))$, and the goal is to seek an approximation $(g_1(u,v), g_2(u,v), g_3(u,v))$. The approximation will be constructed as a superposition of basis functions $B_{jk}(u,v)$

$$g(u,v) = \sum_{l=0}^{p} \sum_{m=0}^{q} c_{lm} B_{lm}(u,v). \tag{1}$$

Formula (1) generalizes the univariate case, e.g., in Newton's form of the global interpolating polynomial

$$p_n(x) = \sum_{i=0}^{n} c_i B_i(x), c_i = [f_0, \ldots, f_i], B_i(x) = \prod_{j=0}^{i-1} (x - x_j). \tag{2}$$

A commonly encountered problem is time evolution of a bounding surface, which can be described through a separation of variables approach as

$$g(t,u,v) = \sum_{l=0}^{p} \sum_{m=0}^{q} c_{lm}(t) B_{lm}(u,v). \tag{3}$$

A simple example is a sphere with time varying radius $r(t)$,

$$f(t,\theta,\varphi) = r(t)(\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi),$$

for which $p = q = 0$, $c_{00}(t) = r(t)$, $B_{00}(\theta,\varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$. A more interesting example is a superposition of spherical harmonics as would occur in the oscillation of a liquid droplet, or the change in an equipotential surface in quantum mechanical descriptions of an atom

$$r(t,\theta,\varphi) = \sum_{l=0}^{p} \sum_{m=l}^{q} c_{lm}(t) Y_l^m(\theta,\varphi), Y_l^m(\theta,\varphi) = e^{im\varphi} P_l^m(\cos\theta),$$

with $P_l^m(\cos\theta)$ denoting the associated Legendre polynomials defined in terms of the ordinary Legendre polynomials $P_l(x)$ by

$$P_l^m(\cos\theta) = (-1)^m (1-x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_l(x).$$

This homework will investigate ways of approximating the spherical harmonics $Y_l^m(\theta, \varphi)$, in conjunction with the univariate approximation of univariate functions $c_{lm}(t)$. The theoretical exercises build up the foundation of construction of orthogonal polynomial bases. The computational part uses these results to approximate the spherical harmonics, and produce physically relevant preditions.

2 Theoretical exercises

1. K&C, 6.8.1, p.404
2. K&C, 6.8.2, p.404 (include a plot comparing exact function and approximations from exercises 1 and 2)
3. K&C, 6.8.5, p.404
4. K&C, 6.8.21, p.405
5. K&C, 6.9.8, p.420. Use both elementary calculus and application of Chebyshev alternation theorem
6. K&C, 6.10.17, p. 438.

3 Computational application

The following tasks will be fully formulated and partially solved in classroom work.

1. Generate data. Choose p, $q \in \{2, 3, 4, 5, 6\}$, and coefficients $c_{lm}(t) = a_{lm} \sin(\omega_{lm}\, t)$, a_{lm} random numbers uniformly distributed in [0,1], $\omega_{lm} = 2\pi(l+m)$, $l = 0, \ldots, p$, $m = 0, \ldots, q$. Compute the positions of points on the surface at times $t_l = l\delta t$, $\delta t = 1/N$, $N = 32$, $l = 0, 1, \ldots, N$, and angles $\theta_j = 2\pi j/P$, $\varphi_k = \pi k/P$, $P = 18$.
2. Investigate recovery of time-varying coefficients by projection onto the spherical harmonic basis.
3. Construct a piecewise linear approximation of the spherical harmonics.
4. Investigate approximation of the time-varying coefficients by projection of the data onto the piecewise linear basis. Compare to exact coefficients.