MATH661 Homework 5 - Approximate solution of differential systems

Posted: Oct 27, Due: 11:55PM, Nov 10

1 Problem statement

Solution of systems of differential equations is a bedrock of scientific computing. Our current understanding of physics is stated as conservation laws: some change occurs due to certain effects. A familiar example is Newton's second law of dynamics, commonly stated as $\boldsymbol{F} = m\boldsymbol{a}$, but more properly formulated as

$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \frac{\mathrm{d}(m\boldsymbol{v})}{\mathrm{d}t} = \boldsymbol{F},$$

or that changes in momentum $\mathrm{d}(m\boldsymbol{v})$ occur due to the action of force over a time interval, $\boldsymbol{F}\,\mathrm{d}t$.

This homework guides you through the basic theoretical concepts, and then applies these to solve a system of ordinary differential equations (ODEs) that arise from the semi-discretization of a partial differential equation (PDE).

2 Theoretical exercises

1. K&C, 8.1.12, 8.1.13, 8.1.19 pp. 528-529.
2. K&C, 8.2.9, p. 538. (Use a symbolic computation package to compute the derivatives needed in the Taylor-series method)
3. K&C, 8.3.4, p. 546
4. K&C, 8.3.5, p. 546
5. K&C, 8.4.4 and 8.4.5, p. 555
6. K&C, 8.4.12, p. 556

3 Implementation and analysis

Consider the following PDE, initial and boundary conditions that define $u(t, x)$, u: $[0, T] \times [0, \pi] \rightarrow \mathbb{R}$,

$$\begin{array}{l} \dfrac{\partial u}{\partial t} = \nabla^2 u = \dfrac{\partial^2 u}{\partial x^2}, \\ u(t, x=0) = 0, u(t, x=\pi) = 0, \\ u(t=0, x) = \sin(x) \exp\left[-2\left(x - \dfrac{\pi}{2}\right)^2\right]. \end{array} \quad (1)$$

1. Obtain a system of ODEs from (1) by introducing the functions $U_k(t) = u(t, x_k)$, $x_k = kh$, $h = \pi/m$, and replacing the x-derivative by a centered finite difference approximation

$$\frac{\partial^2 u}{\partial x^2} = \frac{U_{k+1}(t) - 2U_k(t) + U_{k-1}(t)}{h^2}.$$

Write the resulting system

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{U} = \boldsymbol{A}\boldsymbol{U}, \boldsymbol{U}(t) = \left(U_0 \ \ U_1 \ \ \ldots \ \ U_m \right). \tag{2}$$

2. Write the code to solve (2) using the forward Euler method. Use the code to solve the system for $m = 128, 256$. Experiment with various choices of the time step.

3. Repeat using the fourth-order Runge Kutta method.

4. Repeat using the fourth-order Runge-Kutta-Gill method.

Extra credits:

EC3: Repeat using the adaptive Runge-Kutta-Fehlberg method

EC4: Compute the analytical solution using separation of variables. Construct convergence plots, i.e., plots of the log of the norm of the error $\lg\|\boldsymbol{e}\|$ as a function of $\lg(h)$ for all of the methods considered (forward Euler, 3 Runge-Kutta methods).