MATH661 Homework 7 - Stochastic methods

Posted: Nov 25, Due: 11:55PM, Dec 6

1 Problem statement

Stochastic numerical methods solve problems by random sampling. Given the rapid onset of the Holiday Season, this final homework combines some analysis with good mathematical fun. Try to pair up in teams of 2 to solve the exercises. You're asked to make an investment in your mathematical future: a box of stick matches, some graph paper, cake sprinkles, a compass, and a ruler. The items can be purchsed for about $10 with the cost shared among the entire class, or borrowed from household supplies.

2 Theoretical exercises

1. Read the presentation of Buffon's needle from Wikipedia. One team member draws lines at $\sim 3/4$ of the match length, and the other at $\sim 3/2$ the match length. Throw n matches and record line crossings at $n = 10, 20, 40$ (e.g., include photo from cell phone camera, or present a table of recorded line crossings)

2. Estimate π from the above experiment. Present the theoretical analysis and estimate the error for $n = 10, 20, 40$.

3. Read the presentation of Monte Carlo integration from Wikipedia. Draw a circle on graph paper, and throw some cake sprinkles. Count numbers inside/outside the circle. If interested, you can take a photo and use Mathematica to do the counting. Then throw some more and recount.

4. Estimate π from the above experiment using the integral

$$I_\pi = \int_{x^2+y^2 \leqslant 1} H(x, y)\, \mathrm{d}x \mathrm{d}y,$$

with H the Heaviside function. Present an anlysis of the error

5. Place a piece of graph paper horizontally on a rigid board. Draw concentric circles of radii $r_k = 1, 2, 3, ..., ..., 15$ graph paper units. Fill the innermost circles $r_k < R$ for some choice of R with cake sprinkles, then slightly shake the rigid board (quickly give it a vertical jolt). Record the new positions of the cake sprinkles. Repeat for as many times as you have patience (at least 8 times, 30 times is recommended). Estimate the density of cake sprinkles at each radius $\rho(t, r)$ and each time.

6. Read the theoretical background on Feynman-Kac formulas from Wikipedia. The above experiment essentially solves the diffusion problem

$$\begin{aligned} &\rho_t = \nabla^2 \rho = \frac{\mathrm{d}^2 \rho}{\mathrm{d}r^2} + \frac{1}{r} \frac{\mathrm{d}\rho}{\mathrm{d}r} \\ &\rho(t = 0, r < R) = \rho_0 \end{aligned}$$

The cake sprinkles undergo a random walk through the board shaking. This is a tabletop model for what occurs with molecules (the sprinkles) under the influence of thermal excitation (the board shaking). Compare your experimental solution to the theoretical solution obtained by solving the diffusion problem through separation of variables.

3 Realistic optimization problem

Revisit the diffusion problem from Homework 5 that defines $u(t,x)$, $u:[0,T]\times[0,\pi]\to\mathbb{R}$, as solution of

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sum_{i=1}^{n} A_i \exp\left[-\frac{(x-x_i)^2}{(\pi/20)^2}\right] H_i(t),\\
&u(t,x=0)=0, u(t,x=\pi)=0,\\
&u(t=0,x)=\sin(x).
\end{aligned} \qquad (1)$$

This models the evolution of concetration u from initial condition $u(t=0,\,x)$ due to diffusion and sources placed at positions x_i and intensity A_i turned on after $t>0$ ($H_i(t)=H(t-t_i)$ is the Heaviside function). The problem is inspired from cell biology with $[0,\pi]$ the extent of the cell and u the concentration of Ca. Where should additional calcium sources be placed within the cell, and at what intensity should they release calcium such that the resultis a calcium concentration wave $u_0(t,x)=H(x-vt)$, with $v=6$ over time interval $[0,1]$.

In this homework we solve the same problem again using a stochastic approach. Instead of the Runge-Kutta method, solve the problem by using a Feynman-Kac approach and random numbers.