Final Examination

Solve the following problems (4 course points each). Present a brief motivation of your method of solution.

1. The trace of $\boldsymbol{A} \in \mathbb{C}^{m \times m}$, $\boldsymbol{A} = (a_{ij})_{1 \leqslant i,j \leqslant m}$, $i, j \in \mathbb{N}$, is defined as $\operatorname{tr}(\boldsymbol{A}) = \sum_{i=1}^{m} a_{ii}$.
   a) Is $\operatorname{tr}(\boldsymbol{A})$ invariant under similarity transformations?
   b) Consider $\boldsymbol{B} \in \mathbb{C}^{m \times n}$, $\boldsymbol{C} \in \mathbb{C}^{n \times m}$. Prove that $\operatorname{tr}(\boldsymbol{B}\boldsymbol{C}) = \operatorname{tr}(\boldsymbol{C}\boldsymbol{B})$.
   c) Does the matrix equation $\boldsymbol{A}\boldsymbol{X} - \boldsymbol{X}\boldsymbol{A} = \boldsymbol{I}$ have a solution $\boldsymbol{X} \in \mathbb{C}^{m \times m}$?
   d) For $\boldsymbol{A}$ hermitian, express $\operatorname{tr}(\boldsymbol{A})$ and $\operatorname{tr}(\boldsymbol{A}^{-1})$ in terms of eigenvalues of $\boldsymbol{A}$.
2. Consider $\boldsymbol{A} \in \mathbb{C}^{m \times m}$ nonsingular, and $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{C}^m$ such that $1 + \boldsymbol{d}^* \boldsymbol{A}^{-1} \boldsymbol{c} \neq 0$.
   a) Prove the Sherman-Morrison formula
   $$(\boldsymbol{A} + \boldsymbol{c}\boldsymbol{d}^*)^{-1} = \boldsymbol{A}^{-1} - \frac{\boldsymbol{A}^{-1}\boldsymbol{c}\boldsymbol{d}^*\boldsymbol{A}^{-1}}{1 + \boldsymbol{d}^*\boldsymbol{A}^{-1}\boldsymbol{c}}.$$
   b) Comment on the computational utility of the Sherman-Morrison formula.
3. For $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times m}$ prove:
   a) $\|\boldsymbol{A}\boldsymbol{B}\| \leqslant \|\boldsymbol{A}\| \, \|\boldsymbol{B}\|$
   b)
   $$\left\| \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} \right\|_2 = \max\left(\|\boldsymbol{A}\|_2, \|\boldsymbol{B}\|_2\right).$$
4. Consider the following analogy:

Scalars	**Matrices**
Real numbers, $z = \bar{z}$	Hermitian matrices, $\boldsymbol{A} = \boldsymbol{A}^*$
Imaginary numbers, $z = -\bar{z}$	Skew-Hermitian matrices, $\boldsymbol{A} = -\boldsymbol{A}^*$
Unit circle numbers, $z = e^{i\theta}$, $z\,\bar{z} = \bar{z}\,z = 1$	Unitary matrices, $\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^*\boldsymbol{A} = \boldsymbol{I}$

Table 1. Analogy between scalars $z \in \mathbb{C}$ and matrices $\boldsymbol{A} \in \mathbb{C}^{m \times m}$

   a) What is the image of the imaginary axis through the function $f(z) = (1 - z)/(1 + z)$?
   b) For $\boldsymbol{A}$ skew-hermitian, prove that the result of the Cayley transformation
   $$f(\boldsymbol{A}) = (\boldsymbol{I} - \boldsymbol{A})(\boldsymbol{I} + \boldsymbol{A})^{-1},$$
   is unitary.