## FINAL EXAMINATION

Solve the following problems (4 course points each). Present a brief motivation of your method of solution.

1. The trace of  $\mathbf{A} \in \mathbb{C}^{m \times m}$ ,  $\mathbf{A} = (a_{ij})_{1 \leq i,j \leq m}$ ,  $i, j \in \mathbb{N}$ , is defined as  $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{m} a_{ii}$ .

- a) Is  $tr(\mathbf{A})$  invariant under similarity transformations?
- b) Consider  $\boldsymbol{B} \in \mathbb{C}^{m \times n}$ ,  $\boldsymbol{C} \in \mathbb{C}^{n \times m}$ . Prove that  $\operatorname{tr}(\boldsymbol{B}\boldsymbol{C}) = \operatorname{tr}(\boldsymbol{C}\boldsymbol{B})$ .
- c) Does the matrix equation AX XA = I have a solution  $X \in \mathbb{C}^{m \times m}$ ?
- d) For  $\boldsymbol{A}$  hermitian, express  $tr(\boldsymbol{A})$  and  $tr(\boldsymbol{A}^{-1})$  in terms of eigenvalues of  $\boldsymbol{A}$ .
- 2. Consider  $\mathbf{A} \in \mathbb{C}^{m \times m}$  nonsingular, and  $\mathbf{c}, \mathbf{d} \in \mathbb{C}^m$  such that  $1 + \mathbf{d}^* \mathbf{A}^{-1} \mathbf{c} \neq 0$ .
  - a) Prove the Sherman-Morrison formula

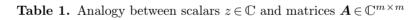
$$(A + cd^*)^{-1} = A^{-1} - \frac{A^{-1}cd^*A^{-1}}{1 + d^*A^{-1}c}$$

- b) Comment on the computational utility of the Sherman-Morrison formula. 3. For  $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times m}$  prove:
  - a)  $\|\boldsymbol{A}\boldsymbol{B}\| \leq \|\boldsymbol{A}\| \|\boldsymbol{B}\|$ b)

$$\left\| \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} \right\|_2 = \max\left( \|\boldsymbol{A}\|_2, \|\boldsymbol{B}\|_2 \right).$$

4. Consider the following analogy:

Scalars	Matrices
Real numbers, $z = \bar{z}$	Hermitian matrices, $\boldsymbol{A} = \boldsymbol{A}^*$
Imaginary numbers, $z = -\bar{z}$	Skew-Hermitian matrices, $\boldsymbol{A} = -\boldsymbol{A}^*$
Unit circle numbers, $z = e^{i\theta}$ , $z \bar{z} = \bar{z} z = 1$	Unitary matrices, $AA^* = A^*A = I$



- a) What is the image of the imaginary axis through the function f(z) = (1-z)/(1+z)?
- b) For A skew-hermitian, prove that the result of the Cayley transformation

$$f(A) = (I - A)(I + A)^{-1},$$

is unitary.