Final Examination (Practice)

Solve the following problems (4 course points each). Present a brief motivation of your method of solution.

1. Prove that $\boldsymbol{A} \in \mathbb{C}^{m\times m}$ can be uniquely factored as $\boldsymbol{A} = \boldsymbol{R}\boldsymbol{U}$, with $\boldsymbol{R}$ hermitian positive definite and $\boldsymbol{U}$ unitary. (This is known as a polar factorization and generalizes the relation $z = re^{i\theta}$)
2. Consider $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{C}^{m\times m}$, both diagonalizable. Prove that $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$ if and only if $\boldsymbol{A}, \boldsymbol{B}$ are simultaneously diagonalizable, i.e., the same invertible matrix $\boldsymbol{P} \in \mathbb{C}^{m\times m}$ leads to $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}_1$, $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \boldsymbol{D}_2$, with $\boldsymbol{D}_1, \boldsymbol{D}_2$ diagonal.
3. Do similar matrices have the same characteristic polynomial?
4. Consider $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ symmetric.
   a) Construct an orthogonal matrix $\boldsymbol{Q}$ to carry out the similarity transformation
   $$\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T = \begin{pmatrix} \lambda & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}.$$
   b) Write pseudo-code that would use the above relation to carry out eigenvalue deflation, e.g., during a QR or Lanczos algorithm.