Homework 1

Due date: Jan 30, 2019, 11:55PM.

Bibliography: Trefethen & Bau, Lectures 1-8. Problems 1-4 = 1 pt each, Problem 5 = 4 points.

1. Exercises 2.3, 2.4, 2.5

2.3. $A \in \mathbb{C}^{m\times m}$, $A = A^*$, $Ax = \lambda x$, $A^* x = \lambda x$, $x \neq 0$.

a) $\lambda \in \mathbb{R}$. Proof: From $Ax = \lambda x$, take adjoint $x^* A^* = x^* A = \bar{\lambda} x^*$. Multiple by x, x^* (non-zero)

$$\begin{array}{lll} Ax = \lambda x & \Rightarrow & x^* A x = \lambda x^* x \\ x^* A = \bar{\lambda} x^* & \Rightarrow & x^* A x = \bar{\lambda} x^* x \end{array} \Rightarrow 0 = (\lambda - \bar{\lambda}) x^* x = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}.$$

b) $Ax = \lambda x, Ay = \mu y, \lambda \neq \mu$, $\lambda, \mu \in \mathbb{R}$

$$\begin{array}{lll} Ax = \lambda x & \Rightarrow & y^* A x = \lambda y^* x \\ y^* A = \mu y^* & \Rightarrow & y^* A x = \mu y^* x \end{array} \Rightarrow 0 = (\lambda - \mu) y^* x = 0 \Rightarrow y^* x = 0 \Rightarrow x \perp y.$$

2.4. $A \in \mathbb{C}^{m\times m}$, $AA^* = A^* A = I$. Take adjoint of eigenvalue relation $AV = \Lambda V$ to obtain $V^* A^* = V^* \Lambda^*$. Multiply the two equations to obtain $V^* A^* A V = V^* \Lambda^* \Lambda V \Rightarrow V^* |\Lambda|^2 V = V^* V$, hence eigenvalues satisfy $|\lambda_i|^2 = 1$, i.e., lie on the unit circle.

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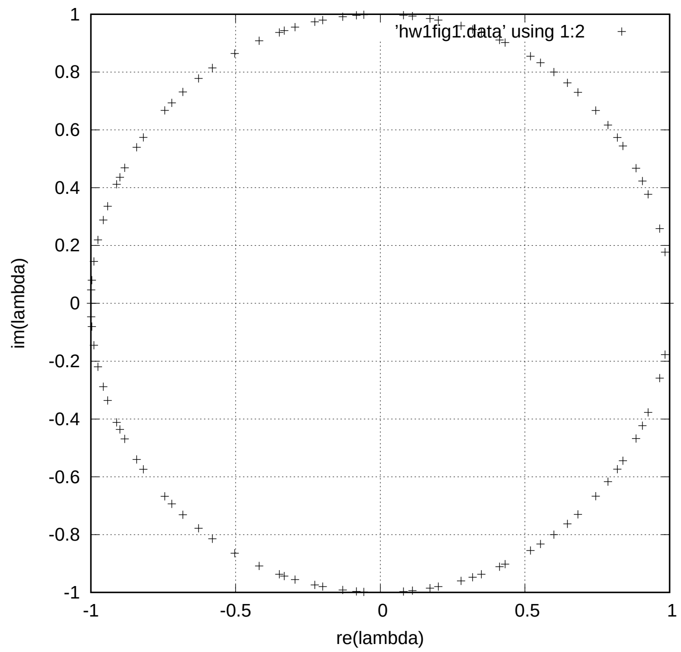
octave: A=randn(100); [Q,R]=qr(A); L=eig(Q);
octave: cd('/home/student/courses/MATH662')
octave: data = [real(L) imag(L)];
octave: save -ascii hw1fig1.data data
octave: [min(abs(L)) max(abs(L))]
  ans =

     1.00000   1.00000

octave:
```

```
GNUplot] cd '/home/student/courses/MATH662';
         set xlabel 're(\lambda)'; set ylabel 'im(\lambda)';
         set grid; set size square;
         plot 'hw1fig1.data' using 1:2 w p
```




```
GNUplot]
```

2.5. $S \in \mathbb{C}^{m \times m}, S^* = -S$

a) As in 2.3

$$\begin{array}{lll} Ax = \lambda x & \Rightarrow & x^* A x = \lambda x^* x \\ -x^* A = \bar{\lambda} x^* & \Rightarrow & -x^* A x = \bar{\lambda} x^* x \end{array} \Rightarrow 0 = (\lambda + \bar{\lambda}) x^* x = 0 \Rightarrow \lambda = -\bar{\lambda} \Rightarrow \lambda = i\alpha, \alpha \in \mathbb{R}.$$

b) This is a matrix generalization of $|1 + i\alpha| > 0$ for any $\alpha \in \mathbb{R}$. Recall that A singular is equivalent to A having a zero eigenvalue. Ask: can $\mu = 0$ be an eigenvalue of $I - S$? If so $(I - S)x = \mu x = 0 \Rightarrow Sx = 1 \cdot x$, implying $1 \in \mathbb{R}$ is an eigenvalue of S, contradicting finding from (a). Or, consider x to be an eigenvector of S, $Sx = i\alpha x$, and compute

$$(I - S)x = x - i\alpha x = (1 - i\alpha)x,$$

hence $1 - i\alpha$ is an eigenvalue of $I - S$, and $|1 - i\alpha| = \sqrt{1 + \alpha^2} > 0$. Note that the above also implies

$$(I - S)^{-1} x = \frac{1}{1 - i\alpha} x,$$

hence $(1 - i\alpha)^{-1}$ is an eigenvalue of $(I - S)^{-1}$.

c) Per (2.4) eigenvalues of a unitary matrix lie on the unit circle. As above, let x be an eigenvector of S, $Sx = i\alpha x$,

$$Qx = (I - S)^{-1}(I + S)x = (I - S)^{-1}(1 + i\alpha)x = (1 + i\alpha)(I - S)^{-1}x = \frac{1 + i\alpha}{1 - i\alpha} x = \mu x,$$

and $|\mu| = |1 + i\alpha| / |1 - i\alpha| = 1$, hence Q unitary.

2. Exercises 3.1-3.6

3.1. $W \in \mathbb{C}^{m \times m}$ nonsingular, $\|x\|_W = \|Wx\|$. Prove norm properties:

1. $\|x\|_W \geqslant 0$ results from $\|Wx\| \geqslant 0$. Consider $\|x\|_W = 0$, or $\|Wx\| = 0 \Rightarrow Wx = 0$. Since W nonsingular cannot have a zero eigenvalue, it results that $x = 0$.

2. $\|x+y\|_W = \|Wx+Wy\| \leqslant \|Wx\| + \|Wy\| = \|x\|_W + \|y\|_W$
3. $\|\alpha x\|_W = \|\alpha Wx\| = |\alpha|\,\|Wx\| = |\alpha|\,\|x\|_W$.

3.2. Write eigenvalue relation for $A \in \mathbb{C}^{m\times m}$, $Ax_i = \lambda_i x_i$, and $\|Ax_i\| = |\lambda_i|\,\|x_i\|$. By definition of induced norm

$$\|A\| = \sup_{x\in\mathbb{C}^m, x\neq 0} \frac{\|Ax\|}{\|x\|} \geqslant \max_i |\lambda_i| = \rho(A).$$

3.3.

a) $\|x\|_\infty = \max_i |x_i| \leqslant (x_1^2 + \cdots + x_m^2)^{1/2} = \|x\|_2$. Equality attained for $x = e_i$, inequality for $x = e_1 + e_2$.

b) $\|x\|_2 = (x_1^2 + \cdots + x_m^2)^{1/2} \leqslant \max_i |x_i| \sqrt{m} = \sqrt{m}\,\|x\|_\infty$. Equality for $x = (1,1,\ldots,1)$, inequality for $x = e_i$.

c) From (a) $\|Ax\|_\infty \leqslant \|Ax\|_2$, and from (b) $1/\|x\|_\infty \leqslant \sqrt{n}/\|x\|_2$, combine to obtain

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \leqslant \sqrt{n}\,\frac{\|Ax\|_2}{\|x\|_2}$$

3. Exercise 5.4. State and solve the analogous problem for skew-symmetric matrices.
4. Exercises 6.1-3, 6.5
5. Consider a black and white image represented as a matrix $A \in \{0,1\}^{32\times 256}$. In each 32×32 block set element values to represent a letter of the words: "absolute", "computer", "measures".

   Read in an image

```
octave> chdir('/home/student/courses/MATH662')
   /home/student/courses/MATH662
octave> load('b.mat');
octave> [U,S,V]=svd(B);
   'Expression1' undefined near line 1 column 7
octave>
```

a) Guess the rank of the matrices. Then, compute the rank of the matrices.

b) Obtain a sequence of approximations A_ν for $\nu = 2^p$, $p = 2,3,4,5$, with A_ν the successive approximations from truncation of the SVD rank-1 expansions.

c) Consider $B(x_1,x_2,x_3) = x_1A_1 + x_2A_2 + x_3A_3$, with $x_1 + x_2 + x_3 = 1$. Repeat (b) for x_1, x_2, x_3 by sampling points within the tetrahedron. Comment.

d) Consider $H(\xi) = A_1 + \xi(A_2 - A_1)$. Repeat (b) for $\xi \in (0,1)$. Comment.